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# Dynamics of higher spin fields and tensorial space

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Abstract: The structure and the dynamics of massless higher spin fields in various dimensions are reviewed with an emphasis on conformally invariant higher spin fields. We show that in $D = 3, 4, 6$ and 10 dimensional space-time the conformal higher spin fields constitute the quantum spectrum of a twistor-like particle propagating in tensorial spaces of corresponding dimensions. We give a detailed analysis of the field equations of the model and establish their relation with known formulations of free higher spin field theory.

Keywords: Field Theories in Higher Dimensions, Space-Time Symmetries, Conformal and W Symmetry, Gauge Symmetry.

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## Contents





## 1. Introduction

The twistor-like particles propagating in tensorial (super)spaces put forward in [1, 2] have the interesting property of being related to massless higher-spin fields. A key point in this construction is the extension of the conventional $D$-dimensional space-time, parametrized by coordinates $x^m$, with extra directions parametrized by antisymmetric tensor coordinates $y^{mn\cdots q}$. In these models the tensorial coordinates correspond to the helicity degrees of freedom of the quantum states of the system in ordinary space-time.

In particular, the $\mathrm{Sp}(8,\mathbb{R})$-invariant twistor superparticle model produces upon quantization an infinite tower of higher-spin fields in $D=4$ space-time, where each and every massless representation of the conformal group $\mathrm{SU}(2,2) \subset \mathrm{Sp}(8,\mathbb{R})$ appears only once. Thus this model turns out to be a realization of a Kaluza-Klein-like mechanism conjectured by Fronsdal [3]. Upon performing an appropriate Fourier transform in twistor space and integrating out the extra tensorial variables $y^{mn}$ one finds that [4]–[7] the wave functions of these higher-spin fields satisfy the unfolded higher spin field equations, which are known to be an appropriate framework in which a self-consistent interaction of higher spin fields can be introduced (see [8] for references and [9] for recent progress).

A (bosonic) tensorial space is parametrized by symmetric $n \times n$ matrix coordinates $X^{\alpha\beta} = X^{\beta\alpha}$ $(\alpha, \beta = 1, \ldots, n)$ linearly transformed by the group $\mathrm{GL}(n, \mathbb{R})$. The dimension of such a space is $\frac{n(n+1)}{2}$. For appropriate *even* values of $n$ the link with ordinary $D$-dimensional space-time coordinates $X^m$ is made by decomposing $X^{\alpha\beta}$ in a basis of symmetric $n \times n$ gamma-matrices

$$X^{\alpha\beta} = x^m \gamma_m^{\alpha\beta} + y^{mn\cdots q}\gamma^{\alpha\beta}_{mn\cdots q}\,, \quad (m = 0, 1, \ldots, D-1; \quad \alpha, \beta = 1, \cdots, n)\,, \tag{1.1}$$

with completely antisymmetric $y^{mn\cdots q}$.

Physically interesting examples of tensorial spaces are

- $n = 2$. The tensorial space has dimension 3 and corresponds to a conventional $D = 3$ space-time without extra $y$-coordinates;
- $n = 4$. The tensorial space is 10-dimensional and corresponds to $D = 4$ space-time enlarged with 6 extra coordinates $y^{mn}$;
- $n = 8$. The tensorial space has dimension 36. It is parametrized by the coordinates $x^m$ of $D = 6$ space-time and by the $3\frac{1}{2}\binom{6}{3} = 30$ components of an SO(3) triplet of anti-self-dual coordinates $y_I^{mnp}$ (I=1,2,3) (where $\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}$);
- $n = 16$. The tensorial space has dimension 136. It is parametrized by the coordinates $x^m$ of $D = 10$ space-time and by $\frac{1}{2}\binom{10}{5} = 126$ anti-self-dual coordinates $y^{mnlpq}$.

For all these cases the space-time dimension $D$ is related to the (spinor) dimension $n$ by the formula $n = 2(D-2)$.

One can also consider tensorial spaces with Grassmann directions parametrized by coordinates $\theta_i^\alpha$ $(i = 1, \ldots, N;\ \alpha, \beta = 1, \cdots, n)$, thus dealing with tensorial superspaces (as $\Sigma^{(\frac{n(n+1)}{2}|n)}$ for $N = 1$ [10]). A physically interesting example is provided by the supergroup manifolds $\mathrm{OSp}(N|n)$.

The $n = 2$ case is well known and corresponds to conventional field theories in $D = 3$ space-time. The physical $D = 4$ space-time higher spin contents of the $n = 4$ model has been studied in detail in [2] and [4]–[7], while only generic properties of higher dimensional generalizations of these models and their generalized (super)conformal structure have been discussed [2], [4]–[6], [11]–[13] and [8]. Though the $n = 8$ and 16 tensorial superparticles were quantized in [2], no detailed analysis of the corresponding spectra of higher spin fields and of their equations of motion in $D = 6$ and 10 space-time, which should follow from the tensorial field equations, has been carried out so far. So the main purpose of this paper is to consider in detail the physical $D = 6$ and $D = 10$ space-time contents of the $n = 8$ and $n = 16$ tensorial models. We will show that, analogously to the $n = 4$, $D = 4$ case, the first quantization of the tensorial particle produces a representation of $\mathrm{Sp}(2n, \mathbb{R})$ which decomposes into an infinite sum of irreducible representations (irreps) of the conformal group $Spin(2, D) \subset \mathrm{Sp}(2n, \mathbb{R})$. In addition to scalar and spinor fields the infinite sets of $D = 6$ and $D = 10$ fields associated with these representations consist of massless higher spin fields of mixed symmetry whose field strengths (or curvatures)[1] are self-dual.

We shall show that the space-time wave equations for the conformal higher spin fields in $D = 4, 6$ and 10 obtained from a scalar and a spinor field equation in the corresponding $n = 2(D-2)$ tensorial space are 'geometric' in the sense that they are written in terms of gauge invariant linearized curvature tensors. In $D = 4$ these higher spin field curvatures are a straightforward generalization of the linearized Riemann curvature and satisfy the same cyclic first and second Bianchi identities as the latter. The higher spin 'Riemann' curvatures are related to the generalized curvatures introduced by de Wit and Freedman [14] via an appropriate (anti)symmetrization of indices. In $D = 4$ Minkowski space-time the free higher spin field equations have been known for a long time, since the paper by Dirac [15]. In the form presently known as 'Bargmann-Wigner equations' they were analyzed from a group-theoretical point of view in [16]. In the massless case these are the 'geometric' equations for higher spin curvatures.[2]. As was shown in [19], the Bargmann-Wigner form of the equations for the massless $D = 4$ higher spin fields is the most convenient one to exhibit their conformal invariance. Geometric free field equations were written by Vasiliev [20, 9] in a moving frame-like (or vielbein-like) formulation for completely symmetric gauge fields propagating in $AdS_D$ space-time. In a metric-like formulation geometric free field equations for arbitrary higher spin gauge fields in flat space-time of any dimension were proposed in [21, 22]. These are a generalization of the $D = 4$ Bargmann-Wigner equations for the higher spin field curvatures. All these equations have a common drawback, namely, that for higher spin fields they cannot be directly obtained from an action principle. Note that the free action for arbitrary higher spin fields constructed in [21] is quadratic in derivatives of the integer spin field potentials and is of the first order in the derivatives of the half-integer field potentials. In the case of symmetric tensor fields it reproduces the Fronsdal [24] and Fang-Fronsdal [25] actions (see [26] for more details). Such actions give rise to the second- or first-order differential equations for the higher spin field *potentials* which (except for the spins $s = 1, 3/2$ and 2) cannot be directly rewritten in terms of higher spin *curvatures* because of the following reason.

[1] In what follows we shall freely use either the name 'field strength' or 'curvature'.

[2] For a pedagogical review, see [17, chapter 1 and section 6.9], and [18] for general relativistic $D = 4$ wave equations and historical references.

A *spin-s curvature* (or *field strength*) is obtained by taking $[s]$ curls of the corresponding *gauge field potential* (the bracket denotes integer part). Thus, by definition, a local free higher spin field equation formulated in terms of the field strength must contain at least $[s]$ partial derivatives. As a result, for $s > 2$ the local geometric higher spin equations contain more than two derivatives and if they were lagrangian the corresponding actions would be of $[s]$-th order in derivatives. So naively the geometric formulations of higher spin theories seem to suffer from the higher-derivative problem which states that free theories whose physical degrees of freedom obey differential equations of order strictly greater than two for bosons, and one for fermions, have ghosts [23]. However, the higher spin fields circumvent this problem in a rather subtle way: the higher spin field strength equations reduce to second or first order differential equations for the corresponding integer or half integer higher spin potentials.

As we have already mentioned, local second and first order differential equations for massless bosonic and fermionic higher spin fields described by symmetric (spinor)-tensors and corresponding actions were constructed in [24, 25] and for generic higher spin fields in [21, 26]. In such formulations the higher spin gauge fields and the gauge parameters satisfy algebraic (trace) constraints. These restrictions on the higher spin gauge fields and parameters look unnatural and, basically, two ways of removing them have been proposed (see [27] for recent reviews of problems of higher spin field theory).

One way is to renounce the locality of the theory. Non-local actions for unconstrained higher-spin gauge fields leading to non-local geometric field equations were constructed by Francia and Sagnotti [28] and generalized to mixed symmetry fields in [29].

The second way of relaxing the trace constraints, keeping locality at the same time, is by introducing a new field called 'compensator' [30, 31]. The resulting field equations are non-lagrangian in the sense that using only the proper higher spin gauge fields and the compensator one is not able to construct an action from which these equations follow. In order to construct a lagrangian one has to introduce extra auxiliary fields [32, 31]. The number of these auxiliary fields increases with the value of the spin of the 'basic' field.

The generalized cohomologies introduced in [33, 34] (and extended to mixed symmetry fields in [22]) further clarified the geometrical structure of higher spin gauge theories. We will see that, for example, the unconstrained gauge invariance and the Bianchi identities of the spin $s$ field strength (with $s > \frac{1}{2}$) are elegantly summarized in terms of the generalized nilpotency of an exterior derivative $\partial$, $\partial^{[s]+1} = 0$ that leads one to the introduction of a generalized cohomology.[3] The cohomological results of [33, 34] are crucial for the possibility of relating the geometric curvature equations to the compensator equations via a generalization of the spin 3 Damour-Deser identity [35]. The latter establishes the relationship between the trace of the spin 3 field curvature with a curl of the kinetic operator acting on the spin 3 field potential in the Fronsdal formulation [24]. For bosons with $s > 2$ this identity expresses $(s-2)$ curls of the spin $s$ Fronsdal kinetic operator as the trace of the field strength [35, 36]. For fermions with $s > 3/2$, it expresses $(s - 3/2)$ curls of the spin

[3]To avoid confusion with the standard exterior derivative $d$ ($d^2 = 0$), the exterior derivative obeying the higher order nilpotency condition is denoted by $\partial$.

$s$ Fang-Fronsdal kinetic operator as the gamma-trace of the field strength. The explicit relationship between the higher spin curvature equations and the compensator equations was shown in [36] for integer spin fields and will be extended to half integer spin fields in this paper. This clarifies how local geometric equations of order $[s]$ are equivalent to (Fang)-Fronsdal equations, and thus explains how higher spin gauge fields circumvent the higher-derivative problem.

The paper is organized as follows. The geometric formulation of higher spin field theory in terms of the generalized curvatures and its application to the description of conformally invariant higher spin fields are discussed in section 2. In section 3 we briefly review the twistor-like particle model in tensorial space with a generic $n$. Section 4 is devoted to the study of the quantum spectrum of the particle in a tensorial space with $n = 2(D-2)$ and $D = 3, 4, 6$ and 10. It is shown that this spectrum consists of an (infinite) set of $D = 3, 4, 6$ and 10 massless (higher spin gauge) fields obeying geometric equations on their curvatures invariant under the generalized conformal group $\mathrm{OSp}(1|4(D-2)) \supset \mathrm{SO}(2, D)$. The relation of these equations to those of Francia and Sagnotti, and of Fang and Fronsdal generalized to mixed symmetry fields is demonstrated. For clarity, we first review the well known $D = 3$ and $D = 4$ cases in subsections 4.1 and 4.2. Then in section 4.3 we pass to the simpler $D = 10$ case, and devote section 4.4 to the technically more involved $D = 6$ case demonstrating that in both of them the spectrum consists of massless self-dual higher spin fields whose field strengths satisfy geometric equations. In the Conclusions we discuss several possible directions for future research.

## 2. Massless higher spin fields in any dimension

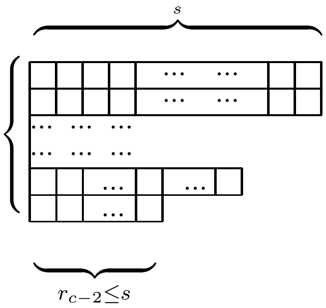
We review here the general properties and equations which (spinor) tensor fields in $D$-dimensional space-time should obey to describe massless higher spin states associated with an appropriate unitary irreducible representation of the Poincaré group. Group-theoretical arguments and the quantum consistency of the theory require that the massless higher spin fields be gauge fields and that their gauge invariant field strengths (or curvatures) satisfy irreducibility conditions which constitute the geometrical higher spin field equations.

### 2.1 Geometric equations

In order to describe a massless unitary irreducible representation of the Poincaré group $\mathrm{ISpin}(1, D-1)$, a (spinor-)tensor field strength should

(i) be *irreducible* with respect to $\mathrm{GL}(D)$. Though, it should be noted that when the spinor structure is introduced, and/or the field strength satisfies a tracelessness condition (which requires the introduction of a metric), the general linear group $\mathrm{GL}(D)$ is restricted to its $\mathrm{SO}(1, D-1)$ subgroup or to $\mathrm{Spin}(1, D-1)$ covering of the latter.

Strictly speaking, the conventional notion of spin (or helicity) is only well defined in $D \leq 4$. In $D > 4$ we will loosely call the spin the number $s$ that characterizes a massless irreducible representation of $\mathrm{ISpin}(1, D-1)$ which corresponds to a Young diagram with $[s]$ columns (where $[s]$ denotes the integer part of $s$). As will become clear in a moment, a spin $s$ field strength should be characterized by the Young diagram[4] $([s],[s],r_1,\ldots,r_{c-2})$




$$\equiv ([s],[s],r_1,\ldots,r_{c-2}) \qquad (2.1)$$

the first two rows of which have an equal length $\ell_1 = \ell_2 = [s]$ (the other rows are arbitrary and $c$ is the length of the first column or number of rows). The field strength tensor is expressed in the antisymmetric basis, in the sense that each column corresponds to a set of antisymmetric indices, e.g. the electromagnetic field strength $F_{mn}$ (s=1) corresponds to the Young diagram $(1,1)$.

Note, however, that setting apart the conformal fields of spin $s$ that are characterized by *rectangular* diagrams (section 2.2), the label $s$ does *not* fully determine the field. Thus, for non-conformal fields the full set of labels in (2.1) is required.

(ii) be *harmonic*, in the sense that it is closed and co-closed (transversal) with respect to each *set* of antisymmetric indices.

If a field strength is closed, then the generalized Poincaré lemmas of [33, 34, 22] imply that the field strength is locally *exact*, i.e. it is equal to $[s]$ curls of a corresponding gauge field potential characterized by the Young diagram $([s],r_1,\ldots,r_{c-2})$ obtained by removing the first row of the field strength diagram. Indeed, the first row of the field strength is made of $[s]$ partial derivatives of the gauge field potential, the symmetry properties making them act as a curl on a given set of antisymmetric indices.

(iii) obey the *Dirac* equation if $s$ is half-integer.

(iv) be *traceless* i.e. $\gamma$-traceless in the case of half-integer $s$ and traceless with respect to any pair of tensorial indices.

Conditions (i), (ii) and (iv) were proposed as field equations for arbitrary bosonic mixed symmetry fields in [22] as a generalization of the $D=4$ Bargmann-Wigner equations [16]. These conditions are not completely independent. For instance, transversality is a consequence of (iii) and of $\gamma$-tracelessness in the fermionic case. In the bosonic case, transversality follows from (i), the closedness condition and tracelessness. Conditions (i) and (iv) insure that the on-shell field strength is irreducible with respect to $\mathrm{Spin}(1,D-1)$. In other words, the linearized curvature is equal to its Weyl part. Moreover, it can be shown that a tensor obeying (i)–(iv) corresponds to a unitary irreducible representation of the

[4] Young diagrams will be denoted by $(\ell_1,\ell_2,\ldots,\ell_c)$. They consist of a finite number $c$ of rows with decreasing numbers of boxes $\ell_1 \geqslant \ell_2 \geqslant \ldots \geqslant \ell_c > 0$.

compact subgroup Spin$(D-2)$ (SO(2) for $D=4$) of the "little group" of Spin$(1,D-1)$. This representation corresponds to a Young diagram $([s], r_1, \ldots, r_{c-2})$ with up to $(D-3)$ rows, so that $c \leq D-2$. In turn, this irrep induces a massless unitary irrep of the Poincaré group ISpin$(1,D-1)$ characterized by a *discrete* spin[5]. This generalization of the $D=4$ Bargmann-Wigner results provides a rigorous proof [37][6] of the fact that the previously proposed equations of motion of a generic mixed symmetry field in flat space-time [39, 40, 22, 29] describe the proper physical higher spin degrees of freedom. Mixed symmetry fields on $AdS$ have been considered in [47] where the situation is more complicated.

Let us note that mixed symmetry fields appear in various physical models. For instance, massive mixed symmetry fields are part of the spectrum of first-quantized strings. One can also obtain mixed symmetry fields by dualizing completely symmetric gauge fields in a space-time of dimension $D>4$ (see for instance [41, 22, 36, 42]). The problem of constructing free field actions for the mixed symmetry fields is efficiently solved in the framework of the BRST approach [43].[7] Alternatively, 'multiform' and 'hyperform' calculus proved to be efficient mathematical tools to deal with the theory of mixed symmetry fields [33, 34, 22, 29].

After this group-theoretical introduction let us make contact with the unconstrained non-local formulation of [28, 30, 31]. The first two conditions (i)–(ii) allow us to express the field strength in terms of a gauge field potential. The field strength is automatically invariant under gauge transformations with unconstrained gauge parameters. Now we have to distinguish among integer spin fields (bosons) and half-integer spin fields (fermions).

For *bosons*, upon solving for (i) and (ii) in terms of the gauge potential the remaining condition on the field strength is its tracelessness. The trace of the field strength with respect to indices belonging to the first two columns is equal to $s-2$ curls of the Labastida kinetic operator [39] which generalizes the Fronsdal kinetic operator to mixed symmetry fields. The tracelessness of the field strength thus states that the Labastida kinetic operator is $\partial^{s-2}$ closed. Then the generalized Poincaré lemma implies that (since for the spin $s$ fields $\partial^{s+1}=0$, and $s+1-(s-2)=3$) the Labastida kinetic operator is $\partial^3$-exact (for $s>2$) in the sense that it is equal to a sum of compensator fields differentiated three times. Each of these compensators corresponds to a Young diagram obtained by removing from the diagram $(s, r_1, \ldots, r_{c-2})$ three cells in different columns. In the case of the symmetric higher spin fields only one compensator field appears, and the resulting form of the higher spin equations coincides with that of the compensator equations of [30, 31]. This relation between the geometric equations on the higher spin curvature and the compensator equations was demonstrated in [36, 37] for an arbitrary bosonic mixed

[5]Note that, as in $D=4$, by restricting the little group to its compact subgroup we remove from the consideration the unphysical massless 'continuous' spin representations.

[6]For a simple explicit example see [38, section 2 and appendix A].

[7]Originally inspired by string field theory, the BRST approach has been used for the analysis of the higher spin spectra of string states in the tensionless limit [44, 31, 45]. Recently, a first-order 'parent' field theory was constructed along the lines of the BRST approach in the context of higher-spin gauge theories [46].

symmetry field. The compensator fields can be gauged away by fixing the traces of the gauge parameters. Alternatively, they can be expressed in a non-local way in terms of the kinetic operators thereby producing non-local equations of [28, 29].

In the *fermionic* case of the half integer spin fields, the relationship between the higher derivative geometric equations and the first order differential equations for the spinor-tensor gauge field potentials has not been considered in the literature before, and here we fill this gap. The reasoning follows the same lines as in the case of the integer spin fields. One can explicitly check that the $\gamma$-trace of the field strength with respect to the indices belonging to the first two columns is equal to $s-3/2$ curls of the Labastida fermionic kinetic operator [40] which generalizes the Fang-Fronsdal kinetic operator to mixed symmetry tensor-spinor fields. The $\gamma$-tracelessness of the field strength which plays the role of the geometric generalization of the Dirac equation thus states that the Labastida kinetic operator is $\partial^{s-3/2}$-closed. Then (since for half integer spin $s$ fields $\partial^{s+1/2}=0$, and $s+\frac{1}{2}-(s-\frac{3}{2})=2$) the generalized Poincaré lemma implies that the Labastida kinetic operator is $\partial^2$-exact (for $s>3/2$) in the sense that it is equal to a sum of fermionic compensator fields differentiated twice. Each of these compensators corresponds to a Young diagram obtained by removing from the diagram $(s-1/2, r_1, \dots, r_{c-2})$ two cells in different columns. In the case of the symmetric tensor-spinor fields only one compensator field appears, and the resulting form of the higher spin equations coincides with that of the compensator equations of [31]. The compensator fields can be gauged away by fixing the $\gamma$-traces of the gauge parameters. Alternatively, they can be expressed in a non-local way in terms of the kinetic operators thereby producing the non-local equations of [28, 29].

We shall now consider in more detail how these general considerations work in the case of conformally invariant higher spin fields.

### 2.2 Conformal higher-spin fields in various dimensions

Among the massless (free) higher spin fields there is an interesting and important subclass of fields whose (at least linearized) equations of motion are conformally invariant. Note, however, that in (anti) de Sitter spaces also partially massless higher spin fields can be conformally invariant (see [48] and references therein).

In $D\geqslant 3$ the necessary and sufficient condition for a unitary irreducible representation of the Poincaré group $\mathrm{ISpin}(1,D-1)$ to be extendable to a unitary irrep of the conformal group $\mathrm{Spin}(2,D)$ is that the representation is induced from the restriction of the 'massless little group' to its compact subgroup $\mathrm{Spin}(D-2)$, and corresponds to a rectangular Young diagram with columns of length equal to $D/2$ [49]. Physically speaking, this means that free conformal unitary field theories are described by massless fields with discrete helicity whose field strength is *chiral*, i.e. self-dual or anti-self-dual. For the odd space-time dimensions this leaves the scalar field and the Dirac fermion as the only possibilities.[8] For even space-time dimensions $D=2c$, the spinorial index should be Weyl and the field should be self-dual (or anti-self-dual) with respect to each set of antisymmetric indices (with the same chirality

[8]Conformal higher spin field theories in $D=3$, considered e.g. in [50], are of a higher-derivative Chern-Simons type and hence do not have propagating physical degrees of freedom.

for all of them).[9] This leads to the field strengths with symmetry properties characterized by *rectangular* Young diagrams $[s] \times D/2$ with columns of equal length $c = D/2$ and rows of equal length $[s]$.[10]

Such self-dual mixed symmetry fields appear in the $(4,0)$ conformal theory in six dimensions, which has been conjectured to be an analogue of the (2,0) conformal theory living on the M5-brane worldvolume [52]. Infinite sets of self-dual higher spin fields also appear in the $D = 6$ and 10 spectrum of states of a first-quantized particle propagating in tensorial spaces with $n = 2(D-2)$, their equations of motion being invariant under the generalized superconformal group $\mathrm{OSp}(1|4(D-2)) \supset \mathrm{SO}(2,D)$. This will be the subject of sections 3 and 4, while below we shall consider the geometrical formulation of the free $D$-dimensional theory of the self-dual higher spin fields characterized by the rectangular Young tableaux $[s] \times D/2$ along the general lines discussed in the previous Subsection.

To simplify a bit notation let us introduce a cumulative index

$$\left[\frac{D}{2}\right] := \left[m_1 \cdots m_{\frac{D}{2}}\right], \tag{2.2}$$

which stands for $\frac{D}{2}$ antisymmetrized indices. Wherever several cumulative indices appear, they will denote the corresponding groups of antisymmetrized indices, e.g.

$$\left[\frac{D}{2} - 1\right]_1 \left[\frac{D}{2}\right]_2 := [m_1 \cdots m_{\frac{D}{2}-1}], [n_1 \cdots n_{\frac{D}{2}}]\,. \tag{2.3}$$

Two cumulative indices which denote the same number of antisymmetric indices are assumed to be symmetric, e.g.

$$\left[\frac{D}{2}\right]_1 \left[\frac{D}{2}\right]_2 = \left[\frac{D}{2}\right]_2 \left[\frac{D}{2}\right]_1 \,. \tag{2.4}$$

Whenever it is unavoidable, we shall use conventional and cumulative indices together.

To illustrate our notation let us recall the familiar example of four-dimensional linearized gravity. In this case $D = 4$ and $s = 2$, and the Riemann tensor is denoted as

$$R_{m_1 m_2,\, n_1 n_2}(x) = -R_{m_2 m_1, n_1 n_2}(x) = R_{n_1 n_2,\, m_1 m_2}(x) \equiv R_{[2]_1\,[2]_2}(x) = R_{[2]_2\,[2]_1}(x)\,. \tag{2.5}$$

The cyclic Bianchi identity and the differential Bianchi identity, respectively, imply that

$$R_{[m_1 m_2,\, n_1] n_2} \equiv R_{[3]_1\,[1]_2} = 0\,, \tag{2.6}$$

$$\partial_{[m_3}\, R_{m_1 m_2],\, n_1 n_2} \equiv \partial_1\, R_{[2]_1\,[2]_2} = 0\,, \tag{2.7}$$

where the subscript of $\partial_1$ means that the exterior derivative is antisymmetrized together with the first group $[2]_1 = [m_1 m_2]$ of antisymmetric indices.

[9] In space-times of Lorentz signature with double even dimensions, i.e. $c = 2k,\ D = 4k$, the chiral field strengths are complex.

[10] More general classes of $D$-dimensional conformal fields and equations which include those corresponding to non-unitary field theories have been discussed in [51].

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### 2.2.1 Integer spin fields

With this notation in mind let

$$R_{\left[\frac{D}{2}\right]_1\cdots\left[\frac{D}{2}\right]_s} = R_{m_1\cdots m_{\frac{D}{2}},\cdots,q_1\cdots q_{\frac{D}{2}}} \tag{2.8}$$

be the curvature (or the field strength) of a conformal integer spin $s$ characterized by a rectangular Young diagram with $\frac{D}{2}$ rows and $s$ columns.

The requirement (i) of section 2 then implies that the curvature tensor is symmetric under exchange of any two cumulative indices and that it satisfies the cyclic Bianchi identity

$$R_{\left[\frac{D}{2}+1\right]_1\left[\frac{D}{2}-1\right]_2\left[\frac{D}{2}\right]_3\cdots\left[\frac{D}{2}\right]_s} = 0\,. \tag{2.9}$$

The curvature is closed if it satisfies the differential Bianchi identity

$$\partial_{[m_1}R_{m_2\cdots n_{\frac{D}{2}+1}],\left[\frac{D}{2}\right]_2\cdots\left[\frac{D}{2}\right]_s} = 0\,, \tag{2.10}$$

and is co-closed (or transverse) if

$$\partial^n R_{n\left[\frac{D}{2}-1\right]_1,\left[\frac{D}{2}\right]_2\cdots\left[\frac{D}{2}\right]_s} = 0\,. \tag{2.11}$$

As in the case of $D = 4$ gravity, the Bianchi identity (2.10) can be written as an exterior derivative acting as a curl on one of the groups of antisymmetric indices of the multiform $R_{[\frac{D}{2}]_1\cdots[\frac{D}{2}]_s}$

$$\partial_1\, R_{\left[\frac{D}{2}\right]_1\cdots\left[\frac{D}{2}\right]_s} = 0\,, \tag{2.12}$$

where the subscript of $\partial_1$ means that the exterior derivative index is antisymmetrized together with the first group $\left[\frac{D}{2}\right]_1$ of the antisymmetric indices.

Let us denote in general by

$$\partial_i \equiv 1\otimes\cdots\otimes\partial_{m_i}\otimes\cdots\otimes 1 \quad (i = 1,\cdots,s) \tag{2.13}$$

the exterior derivative (curl) antisymmetrized with the $i$-th group $\left[\frac{D}{2}\right]_i$ of antisymmetric indices.[11] Then,

$$\partial_i\,\partial_j \equiv 1\otimes\cdots\otimes\partial_{m_i}\otimes\cdots\otimes\partial_{m_j}\otimes\cdots\otimes 1 = \partial_j\,\partial_i, \quad (\partial_i)^2 = 0$$

are "curled" with $\left[\frac{D}{2}\right]_i$ and $\left[\frac{D}{2}\right]_j$, etc.

Let us now introduce the differential operator

$$\partial := \sum_{i=1}^{s}\partial_i\,. \tag{2.14}$$

[11] Actually, if we worked with differential *multiforms*, the differential operator (2.13) would correspond to the operator $1\otimes\cdots\otimes d\otimes\cdots\otimes 1 \equiv (1\otimes\cdots\otimes dx^{m_i}\otimes\cdots\otimes 1)\,\partial_{m_i}$, where the exterior derivative $d$ stands in the $i$-th place and acts on the $i$-th block of the *multiform* characterized by the cumulative antisymmetric tensor index $[\frac{D}{2}]_i$. However, since we would like to keep track of the indices, we prefer to use the definition (2.13) where the partial derivative $\partial_{m_i}$ acts as a curl within the $i$-th cumulative index.

In view of the nilpotency of the exterior derivative ($\partial_i^2 = 0$ for each $i$) the differential operator $\partial$ satisfies the higher order nilpotency condition

$$\partial^{s+1} = \partial\, \partial^s = 0\,, \qquad \text{where} \qquad \partial^s := s \prod_{i=1}^{s} \partial_i = s\, \partial_1 \otimes \partial_2 \otimes \cdots \otimes \partial_s\,. \tag{2.15}$$

According to the generalized Poincaré lemma of [34], the Bianchi identity (2.10) implies that (at least locally) the curvature is the $s$-th derivative of a potential, which in the 'multiform' notation reads

$$R_{\left[\frac{D}{2}\right]_1 \cdots \left[\frac{D}{2}\right]_s} = \partial_1 \cdots \partial_s\, \varphi_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_s}\,, \tag{2.16}$$

where, as defined in (2.13), each $\partial_i$ acts as an exterior derivative (curl) on the corresponding group $\left[\frac{D}{2}-1\right]_i$ of antisymmetric indices. Using the notation (2.15), eq. (2.16) can be written in a more schematic way as follows

$$R = \frac{1}{s}\, \partial^s\, \varphi\,,$$

which is the generalization to the spin $s$ fields of the well known expression of the electromagnetic field strength in terms of the curl of the spin 1 field potential $F = \partial\, A$.

The field $\varphi \equiv \varphi_{[\frac{D}{2}-1]_1 \cdots [\frac{D}{2}-1]_s}$ is the conformal gauge field potential of integer spin $s$ characterized by the rectangular Young diagram $s \times (\frac{D}{2}-1)$, so it is symmetric under the exchange of any two of the $s$ cumulative indices $[\frac{D}{2}-1]_i$ and satisfies a cyclic Bianchi identity similar to (2.9),

$$\varphi_{\left[\frac{D}{2}\right]_1 \left[\frac{D}{2}-2\right]_2 \left[\frac{D}{2}-1\right]_3 \cdots \left[\frac{D}{2}-1\right]_s} = 0\,.$$

Let us note that de Wit and Freedman [14] constructed curvature tensors out of the $s$ derivatives of the symmetric higher spin gauge fields $\varphi_{m_1 \cdots m_s}(x)$ in an alternative way. Their curvatures have two groups of $s$ symmetric indices and they are symmetric or antisymmetric under the exchange of these groups of indices depending on whether $s$ is even or odd

$$R_{m_1 \cdots m_s,\, n_1 \cdots n_s} = (-1)^s\, R_{n_1 \cdots n_s,\, m_1 \cdots m_s}\,. \tag{2.17}$$

For $D = 4$, the tensor (2.17) is related to the tensor (2.16) by the antisymmetrization of each pair $[m_i,\, n_i]$ of indices of the former. In what follows we will work with the generalized Riemann curvatures.

Due to (2.15), the field strengths (2.16) and (2.17) are invariant under the following gauge transformations of the gauge potential [34]

$$\begin{aligned} \delta\varphi_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_s} &= \partial_1\, \xi_{\left[\frac{D}{2}-2\right]_1 \left[\frac{D}{2}-1\right]_2 \cdots \left[\frac{D}{2}-1\right]_s} + \partial_2\, \xi_{\left[\frac{D}{2}-1\right]_1 \left[\frac{D}{2}-2\right]_2 \cdots \left[\frac{D}{2}-1\right]_s} + \cdots \\ &= \sum_{i=1}^{s} \partial_i\, \xi_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-2\right]_i \cdots \left[\frac{D}{2}-1\right]_s}\,, \end{aligned} \tag{2.18}$$

where $\xi(x)$ is an unconstrained gauge function characterized by the Young diagram $(s, \ldots, s, s-1)$ with $[\frac{D}{2}-1]$ rows.

When the conditions (i) a (ii) of section 2 on the integer spin curvature are resolved in terms of the gauge potential, the only one which remains is (iv), i.e. tracelessness of the curvature tensor in any pair of its indices belonging to different cumulative indices

$$tr\, R_{\left[\frac{D}{2}\right]_1 \cdots \left[\frac{D}{2}\right]_s} = 0\,. \qquad (2.19)$$

This is the field equation that generalizes the linearized Einstein equation $R^p{}_{m,\,pn} = R_{mn} = 0$ for spin 2.

Recall that for conformal fields in even-dimensional space-times, the self-duality condition

$$R_{\left[\frac{D}{2}\right]_1 \cdots \left[\frac{D}{2}\right]_s} = \pm \frac{i^{\frac{D}{2}+1}}{\frac{D}{2}!}\, \epsilon_{\left[\frac{D}{2}\right]_1,\, n_1 \cdots n_{\frac{D}{2}}}\, R^{n_1 \cdots n_{\frac{D}{2}}}{}_{\left[\frac{D}{2}\right]_2 \cdots \left[\frac{D}{2}\right]_s}$$

is the actual field equation because tracelessness and transversality follow from the self-duality condition provided the curvature satisfies the Bianchi identities.

Analyzing the form of the left hand side of eq. (2.19) in terms of the gauge field potential (2.16) one gets the generalization of the spin 3 Damour-Deser identity [35]

$$\mathrm{tr}\, R_{\left[\frac{D}{2}\right]_1 \cdots \left[\frac{D}{2}\right]_s} = \partial_1 \cdots \partial_{s-2}\, G_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_{s-2} \left[\frac{D}{2}-1\right]_{s-1} \left[\frac{D}{2}-1\right]_s}\,, \qquad (2.20)$$

where $G$ is the kinetic operator acting on the gauge field potential [39]

$$\begin{aligned} G_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_s} &= \Box\, \varphi_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_s} - \sum_{i=1}^{s} \partial_i\, \partial^m \varphi_{\left[\frac{D}{2}-1\right]_1, \cdots,\, m\left[\frac{D}{2}-2\right]_i, \cdots \left[\frac{D}{2}-1\right]_s} + \\ &+ \sum_{j>i=1}^{s} \partial_i\, \partial_j\, \eta^{mn}\, \varphi_{\left[\frac{D}{2}-1\right]_1 \cdots,\, m\left[\frac{D}{2}-2\right]_i, \cdots,\, n\left[\frac{D}{2}-2\right]_j, \cdots \left[\frac{D}{2}-1\right]_s}\,, \end{aligned} \qquad (2.21)$$

where (in accordance with our notation and convention) the sums are taken over the terms with the exterior derivative $\partial_i$ indices antisymmetrized with those of the corresponding group $\left[\frac{D}{2} - 2\right]_i$.

When the curvature tensor satisfies the tracelessness condition (2.19) the left hand side of eq. (2.20) vanishes, which implies that the multiform $G$ is $\partial^{s-2}$-closed. In virtue of the generalized Poincaré lemma [34] this means that (at least locally) $G$ is $\partial^3$-exact, i.e. has the form [36]

$$G_{\left[\frac{D}{2}-1\right]_1 \cdots \left[\frac{D}{2}-1\right]_s} = \sum_{k>j>i=1}^{s} \partial_i\, \partial_j\, \partial_k\, \rho_{\left[\frac{D}{2}-1\right]_1 \cdots, \left[\frac{D}{2}-2\right]_i, \cdots, \left[\frac{D}{2}-2\right]_j, \cdots \left[\frac{D}{2}-2\right]_k, \cdots \left[\frac{D}{2}-1\right]_s}\,, \qquad (2.22)$$

where the tensor field $\rho(x)$ is characterized by the Young diagram $(s, \ldots, s, s-3)$ with $[\frac{D}{2} - 1]$ rows. The tensor $\rho(x)$ is called 'compensator' field since its gauge transformation compensates the non-invariance of the kinetic operator $G(x)$ under the unconstrained local variations (2.18) of the gauge field potential $\varphi(x)$. Therefore, eq. (2.22) generalizes to arbitrary rectangular Young diagrams the compensator equation given in [30, 31].



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The gauge variation of $G(x)$ is

$$\delta G_{\left[\frac{D}{2}-1\right]_1\cdots\left[\frac{D}{2}-1\right]_s} = \sum_{k>j>i=1}^{s} \partial_i\,\partial_j\,\partial_k\,\eta^{mn}\,\xi_{\left[\frac{D}{2}-1\right]_1\cdots,\left[\frac{D}{2}-2\right]_i,\cdots,\left[\frac{D}{2}-2\right]_j{}^{m},\cdots\left[\frac{D}{2}-2\right]_k{}^{n},\cdots\left[\frac{D}{2}-1\right]_s} \tag{2.23}$$

and it is compensated by the gauge shift of the field $\rho(x)$ with the trace of the gauge parameter

$$\delta\rho_{\left[\frac{D}{2}-2\right]_1\left[\frac{D}{2}-2\right]_2\left[\frac{D}{2}-2\right]_3\left[\frac{D}{2}-1\right]_4\cdots\left[\frac{D}{2}-1\right]_s} = \eta^{mn}\,\xi_{\left[\frac{D}{2}-2\right]_1{}^{m},\left[\frac{D}{2}-2\right]_2{}^{n},\left[\frac{D}{2}-2\right]_3\left[\frac{D}{2}-1\right]_4\cdots\left[\frac{D}{2}-1\right]_s}\,. \tag{2.24}$$

So the compensator can be gauged away by choosing a gauge parameter $\xi(x)$ with the appropriate trace. Then the equations of motion of the gauge field $\phi(x)$ become the second order differential equations of Labastida, which generalize those of Fronsdal for mixed symmetry fields

$$G_{\left[\frac{D}{2}-1\right]_1\cdots\left[\frac{D}{2}-1\right]_s} = 0\,. \tag{2.25}$$

They are invariant under the gauge transformations (2.18) with traceless multi-index gauge functions $\xi(x)$ and also require the higher spin gauge field to be double traceless.

### 2.2.2 Non-local form of the higher spin equations

We shall now demonstrate how the higher spin field equations with the compensator (2.22) are related to the non-local equations of Francia and Sagnotti [28, 30]. We shall consider the simple (standard) example of a gauge field of spin 3. The case of a generic spin $s$ can be treated in a similar but more tedious way. In a somewhat different way the relation of the compensator equations to non-local higher spin equations was discussed in [30].

For the *spin* 3 field the compensator equation takes the form

$$G_{mnp} := \Box\,\varphi_{mnp} - 3\partial_q\,\partial_{(m}\,\varphi_{np)}{}^{q} + 3\partial_{(m}\,\partial_n\,\varphi_{p)q}{}^{q} = \partial_m\,\partial_n\,\partial_p\,\rho(x)\,, \tag{2.26}$$

where () stands for the symmetrization of the indices with weight one and $\rho(x)$ is the compensator, which is a scalar field in the case of spin 3.

We now take the derivative and then the double trace of the left and the right hand side of this equation and get

$$\partial_m\,G^{mn}{}_{n} = \Box^2\,\rho(x)\,. \tag{2.27}$$

Modulo the *doubly harmonic* zero modes $\rho_0(x)$, satisfying $\Box^2\,\rho_0(x) = 0$, one can solve eq. (2.27) for $\rho(x)$ in a non-local form

$$\rho(x) = \frac{1}{\Box^2}\,\partial_m\,G^{mn}{}_{n}\,. \tag{2.28}$$

Substituting this solution into the spin 3 field equation (2.26) we get one of the forms of non-local equations constructed in [28, 30]

$$G_{mnp} := \Box\,\varphi_{mnp} - 3\partial_q\,\partial_{(m}\,\varphi_{np)}{}^{q} + 3\partial_{(m}\,\partial_n\,\varphi_{p)q}{}^{q} = \frac{1}{\Box^2}\,\partial_m\,\partial_n\,\partial_p\,(\partial_q\,G^{qr}{}_{r})\,. \tag{2.29}$$

Let us now consider the more complicated example of *spin* 4. In the Fronsdal formulation, the fields of spin 4 and higher feature one more restriction: they are double traceless. We shall show how this constraint appears upon gauge fixing the compensator equation, which for the spin 4 field has the form

$$G_{mnpq} := \Box\,\varphi_{mnpq} - 4\partial_r\,\partial_{(m}\,\varphi_{npq)}{}^r + 6\partial_{(m}\,\partial_n\,\varphi_{pq)r}{}^r = 4\partial_{(m}\,\partial_n\,\partial_p\,\rho_{q)}(x)\,. \tag{2.30}$$

Taking the double trace of (2.30) we have

$$G^{mn}{}_{mn} = 3\Box\,\varphi^{mn}{}_{mn} = 4\Box\,\partial_m\,\rho^m\,. \tag{2.31}$$

Taking the divergence and the trace of (2.30) we get

$$\partial_m\,G^{mn}{}_{np} = \Box^2\rho_p + 3\partial_p\,\Box\,\partial_m\,\rho^m = \Box^2\rho_p + \frac{3}{4}\,\partial_p\,G^{mn}{}_{mn}\,, \tag{2.32}$$

where we have used (2.31) to arrive at the right hand side of (2.32).

From (2.32) we find that modulo the zero modes $\rho_0^p$ of $\Box^2\,\rho_0^p = 0$, the compensator field is non-locally expressed in terms of the (double) trace of the Fronsdal kinetic term

$$\rho_p = \frac{1}{\Box^2}\left(\partial_m\,G^{mn}{}_{np} - \frac{3}{4}\,\partial_p\,G^{mn}{}_{mn}\right). \tag{2.33}$$

Inserting (2.33) into (2.30) we get one of the forms of the non-local Francia-Sagnotti equations for the spin 4 field.

Consider now the following identity

$$\partial_q\,G^q{}_{mnp} - \partial_{(m}\,G^q{}_{np)q} = -\frac{3}{2}\partial_m\,\partial_n\,\partial_p\,\varphi^{qr}{}_{qr} = -2\,\partial_m\,\partial_n\,\partial_p\,(\partial_q\,\rho^q)\,. \tag{2.34}$$

From (2.31) and (2.34) it follows that modulo constant, linear and quadratic terms in $x^m$ (which can be put to zero by requiring an appropriate asymptotic (fall-off) behavior of the wave functions at infinity) the double trace of the gauge field $\varphi(x)$ is proportional to the divergence of $\rho_q(x)$

$$\varphi^{mn}{}_{mn} = \frac{4}{3}\,\partial_m\,\rho^m\,. \tag{2.35}$$

Therefore, when we partially fix the gauge symmetry by putting $\rho_q(x) = 0$, the double trace of the gauge field also vanishes and we recover the Fronsdal formulation with the traceless gauge parameter and the double traceless gauge field.

### 2.2.3 Half integer spin fields

Let us generalize the previous consideration to the case of fermions. The fermionic spin-$s$ field strength $\mathcal{R}^\alpha$ is the spinor-tensor

$$\mathcal{R}^\alpha{}_{[\frac{D}{2}]_1\cdots[\frac{D}{2}]_{s-\frac{1}{2}}}(x) \tag{2.36}$$

whose tensorial part is described by the rectangular Young tableau $\frac{D}{2}\times(s-\frac{1}{2})$. It satisfies Bianchi identities analogous to (2.12) and can be expressed, similarly to (2.16), in terms of a multi-index fermionic field potential

$$\mathcal{R}^{\alpha}{}_{[\frac{D}{2}]_1\cdots[\frac{D}{2}]_{s-\frac{1}{2}}} = \partial_1\cdots\partial_{s-\frac{1}{2}}\,\psi^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}}\,, \tag{2.37}$$

where the fermionic conformal gauge field $\psi^{\alpha}(x)$ is the spinor-tensor characterized by the rectangular Young diagram $(\frac{D}{2}-1)\times(s-1/2)$. The gauge transformations of $\psi^{\alpha}(x)$ are similar to (2.18) with the only difference that the gauge parameter $\xi^{\alpha}(x)$ is now a spinor-tensor characterized by the diagram $(s-1/2,\ldots,s-1/2,s-3/2)$ with $(\frac{D}{2}-1)$ rows. The fermionic generalization of the Damour-Deser identity is

$$(\gamma^m\,\mathcal{R})^{\alpha}{}_{m\,[\frac{D}{2}-1]_1,[\frac{D}{2}]_2\cdots[\frac{D}{2}]_{s-\frac{1}{2}}} = \partial_1\cdots\partial_{s-\frac{3}{2}}\,G^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{3}{2}}\,[\frac{D}{2}-1]_{s-\frac{1}{2}}}\,, \tag{2.38}$$

where the fermionic kinetic operator $G^{\alpha}$ acting on the gauge field $\psi^{\alpha}$ is [39]

$$G^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} = \partial\!\!\!/\psi^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} - \sum_{i=1}^{s-\frac{1}{2}}\partial_i\,(\gamma^n\psi)^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots n[\frac{D}{2}-2]_i,\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}}\,. \tag{2.39}$$

The field strength (2.16) is invariant under the following gauge transformations of the gauge potential [34]

$$\delta\psi^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} = \sum_{i=1}^{s}\partial_i\,\xi^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-2]_i\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}}\,. \tag{2.40}$$

When the fermionic field strength satisfies the $\gamma$-tracelessness condition (iv) of section 2, i.e.

$$(\gamma^m\,\mathcal{R})^{\alpha}{}_{m\,[\frac{D}{2}-1]_1,[\frac{D}{2}]_2\cdots[\frac{D}{2}]_{s-\frac{1}{2}}} = 0\,, \tag{2.41}$$

eq. (2.38) implies that $G^{\alpha}$ is $\partial^{s-\frac{3}{2}}$-closed. Since $\partial^{s+\frac{1}{2}}\equiv 0$, by virtue of the generalized Poincaré lemma $G^{\alpha}$ is $\partial^2$-exact

$$G^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} = \sum_{j>i=1}^{s-\frac{1}{2}}\partial_i\,\partial_j\,\rho^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots,[\frac{D}{2}-2]_i,\cdots,[\frac{D}{2}-2]_j,\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}}\,, \tag{2.42}$$

where $\rho^{\alpha}(x)$ is the fermionic compensator characterized by a Young diagram $(s-1/2,\ldots,s-1/2,s-5/2)$ with $(\frac{D}{2}-1)$ rows.

Equation (2.42) is the generalization of the compensator equation given in [31] to arbitrary rectangular Young diagrams. The demonstration of its relation to the gamma-traceless part of the fermionic higher spin field strength is a new result.

The gauge variation of $G^{\alpha}(x)$ is

$$\delta G^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} = \sum_{j>i=1}^{s-\frac{1}{2}}\partial_i\,\partial_j\,(\gamma^m\,\xi)^{\alpha}{}_{[\frac{D}{2}-1]_1\cdots,[\frac{D}{2}-2]_i,\cdots,[\frac{D}{2}-2]_j\,m,\cdots,\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} \tag{2.43}$$

and it is compensated by a gauge shift of the field $\rho^\alpha(x)$ given by the $\gamma$-trace (*i.e.* the contraction of the gamma matrix vector index with one inside a cumulative index) of the gauge parameter

$$\delta\rho^\alpha{}_{[\frac{D}{2}-2]_1\,[\frac{D}{2}-2]_2\,[\frac{D}{2}-1]_3\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} = (\gamma^m\,\xi)^\alpha_{[\frac{D}{2}-2]_1\,m\,,\,[\frac{D}{2}-2]_3\,[\frac{D}{2}-1]_3\cdots[\frac{D}{2}-1]_s}\,. \tag{2.44}$$

Thus, the compensator can be gauged away by choosing a gauge parameter $\xi^\alpha(x)$ with the appropriate $\gamma$-trace. Then, the equations of motion of the gauge field $\psi^\alpha(x)$ become the first order differential equations of Labastida which generalize to mixed symmetry fields those of Fang and Fronsdal

$$\partial\!\!\!/\psi^\alpha_{[\frac{D}{2}-1]_1\cdots[\frac{D}{2}-1]_{s-\frac{1}{2}}} - \sum_{i=1}^{s}\partial_i\,(\gamma^n\psi)^\alpha_{[\frac{D}{2}-1]_1\cdots,\,n[\frac{D}{2}-2]_i\,,\cdots[\frac{D}{2}-1]_s} = 0\,. \tag{2.45}$$

These equations are invariant under the gauge transformations (2.40) with $\gamma$-traceless parameters.

Alternatively, one can get the non-local Francia-Sagnotti equations for fermions by taking a particular non-local solution for the compensator field in terms of the fermionic kinetic operator $G^\alpha$. As a simple example consider the $s = 5/2$ case. Eq. (2.42) takes the form

$$G^\alpha_{mn} := \partial\!\!\!/\psi^\alpha_{mn} - 2\partial_{(m}\,(\gamma^q\psi)^\alpha_{n)q} = \partial_m\,\partial_n\,\rho^\alpha(x)\,. \tag{2.46}$$

Taking the trace of (2.46) we get

$$\Box\,\rho^\alpha = G^{\alpha p}{}_p\,. \tag{2.47}$$

Hence, modulo the zero modes $\rho_0^\alpha(x)$ of the Klein-Gordon operator $\Box\,\rho_0^\alpha = 0$ the compensator field is non-locally expressed in terms of the trace of $G^\alpha_{mn}$

$$\rho^\alpha = \frac{1}{\Box}\,G^{\alpha p}{}_p\,. \tag{2.48}$$

Substituting (2.48) into (2.46) we get the Francia-Sagnotti equation for the fermionic field of spin 5/2

$$G^\alpha_{mn} := \partial\!\!\!/\psi^\alpha_{mn} - 2\partial_{(m}\,(\gamma^q\psi)^\alpha_{n)q} = \frac{1}{\Box}\,\partial_m\,\partial_n\,G^{\alpha p}{}_p\,. \tag{2.49}$$

In the same way one can relate the compensator equations for an arbitrary half integer spin field to the corresponding non-local field equation. As in the bosonic case, one can find that for $s \geq \frac{7}{2}$ the triple-gamma trace of the fermionic gauge field potential is expressed in terms of the $\gamma$-trace and the divergence of the compensator field.

## 3. Dynamics of the tensorial twistor-like particle. Preonic equation and conformally invariant fields

We now show that the conformal integer and half integer higher spin fields in 4-, 6- and 10-dimensional space-time satisfying the geometrical equations considered in the previous Section arise as a result of the quantization of a twistor-like particle propagating, respectively, in the $n = 4$, 8 and 16 tensorial spaces. The quantum spectrum of this particle contains an infinite number of conformal higher spin states. The state of each spin appears only once in the spectrum of $n = 4$ ($D = 4$) and $n = 16$ ($D = 10$) tensorial particles, while (except for the scalar and the spinor state) the higher spin fields in the spectrum of the $n = 8$ tensorial particle form in the corresponding $D = 6$ space-time higher *isospin* representations of an internal group SO(3). Let us however note that the considerations below (up to eq. (3.12)) are valid for arbitrary $n$.

The action proposed in [1] to describe a twistor-like particle propagating in tensorial space has the form

$$S[X, \lambda] = \int E^{\alpha\beta}\left(X(\tau)\right)\, \lambda_\alpha(\tau)\, \lambda_\beta(\tau)\,, \tag{3.1}$$

where $\lambda_\alpha(\tau)$ is an auxiliary commuting real spinor, a *twistor-like* variable, and $E^{\alpha\beta}(x(\tau))$ is the pull back on the particle worldline of the tensorial space vielbein. In this paper we will deal with flat tensorial space. In this case

$$E^{\alpha\beta}(X(\tau)) = d\tau\, \partial_\tau X^{\alpha\beta}\,(\tau) = dX^{\alpha\beta}\,(\tau)\,. \tag{3.2}$$

The dynamics of particles on the supergroup manifolds $\mathrm{OSp}(N|n, \mathbb{R})$ (which are the tensorial extensions of AdS superspaces) was considered for $N = 1$ in [11, 6, 7] and for a generic $N$ in [4, 5]. The twistor-like superparticle in $n = 32$ tensorial superspace was considered in [10] as a point-like model for BPS preons [53], the hypothetical $\frac{31}{32}$-supersymmetric constituents of M-theory.

The action (3.1) is manifestly invariant under global $\mathrm{GL}(n, \mathbb{R})$ transformations. Without going into details which the reader may find in [1, 4, 6], let us note that the action (3.1) is invariant under global $\mathrm{Sp}(2n, \mathbb{R})$ transformations, acting non-linearly on $X^{\alpha\beta}$ and on $\lambda_\alpha$, *i.e.* it possesses the symmetry considered by Fronsdal to be an underlying symmetry of higher spin field theory in the case $n = 4$, $D = 4$ [3].

Applying the hamiltonian analysis to the particle model described by (3.1) and (3.2), one finds that the momentum conjugate to $X^{\alpha\beta}$ is related to the twistor-like variable $\lambda_\alpha$ via the constraint

$$P_{\alpha\beta} = \lambda_\alpha \lambda_\beta\,. \tag{3.3}$$

This expression is the direct analog and generalization of the Cartan-Penrose (twistor) relation for the particle momentum $P_m = \lambda\gamma_m\lambda$. In virtue of the Fierz identity (A.6) the twistor particle momentum is light-like in $D = 3, 4, 6$ and 10 space-time. Therefore, in the tensorial spaces corresponding to these dimensions of space-time the first-quantized particles are massless [1, 2].

The quantum counterpart of (3.3) is the equation [2]

$$D_{\alpha\beta}\Phi(X, \lambda) = \left(\frac{\partial}{\partial X^{\alpha\beta}} - i\lambda_\alpha\lambda_\beta\right)\Phi(X, \lambda) = 0\,, \tag{3.4}$$

where the wave function $\Phi(X, \lambda)$ depends on $X^{\alpha\beta}$ and $\lambda_\alpha$. Eq. (3.4) has been shown to correspond [10] to a BPS preon [53] and thus may be called *preonic equation*. The general solution of (3.4) is the plane wave

$$\Phi(X, \lambda) = e^{iX^{\alpha\beta}\lambda_\alpha\lambda_\beta}\varphi(\lambda)\,, \tag{3.5}$$

where $\varphi(\lambda)$ is a generic function of $\lambda_\alpha$.

One can now Fourier transform the function (3.5) to another representation to be called $Y$-representation

$$C(X,Y) = \int d^4\lambda\, e^{-iY^\alpha \lambda_\alpha} \Phi(X,\lambda) = \int d^4\lambda\, e^{-iY^\alpha\lambda_\alpha + iX^{\alpha\beta}\lambda_\alpha\lambda_\beta} \varphi(\lambda)\,. \tag{3.6}$$

The wave function $C(X,Y)$ satisfies the Fourier transformed preonic equation

$$\left(\frac{\partial}{\partial X^{\alpha\beta}} + i\frac{\partial^2}{\partial Y^\alpha \partial Y^\beta}\right) C(x,Y) = 0\,. \tag{3.7}$$

This equation has been analyzed in detail in [4] for wave functions that are power series in $Y^\alpha$

$$C(X,Y) = \sum_{n=0}^{\infty} C_{\alpha_1\cdots\alpha_n}(X)\, Y^{\alpha_1}\cdots Y^{\alpha_n} = b(X) + f_\alpha(X) Y^\alpha + \cdots\,. \tag{3.8}$$

In view of the Fourier relation (3.6) the series in $Y^\alpha$ naturally arises as a result of the series expansion of the exponent $e^{-iY^\alpha\lambda_\alpha}$. Thus the scalar field $b(X)$ and the spinor field $f_\alpha(X)$ in (3.8) are related to the wave function $\Phi(x,\lambda)$ by the following integral expressions

$$b(X) \,=\, \int d^n\lambda\ \Phi(X,\lambda)\ + \text{c.c.}\,, \tag{3.9}$$

$$f_\alpha(X) \,=\, -i\int d^n\lambda\ \lambda_\alpha\, \Phi(X,\lambda)\ + \text{c.c.}\,, \tag{3.10}$$

where *c.c.* stands for 'complex conjugate' since in what follows we shall deal with the real fields $b(X)$ and $f_\alpha(X)$.

Inserting (3.8) into (3.7) one finds that the scalar field $b(X)$ and the spinor field $f_\alpha(X)$ must satisfy the following equations found in [4]

$$\partial_{\alpha\beta}\partial_{\gamma\delta}\, b(X) - \partial_{\alpha\gamma}\partial_{\beta\delta}\, b(X) \,=\, 0\,, \tag{3.11}$$

$$\partial_{\alpha\beta} f_\gamma(X) - \partial_{\alpha\gamma} f_\beta(X) \,=\, 0\,. \tag{3.12}$$

These fields are dynamical, while all higher components in the expansion (3.8) are expressed in terms of (higher) derivatives of the basic fields $b(X)$ and $f_\alpha(X)$ and, hence, are auxiliary fields [4]. In [4] it was also shown that eqs. (3.11) and (3.12) are invariant under the generalized superconformal transformations generating the supergroup $\mathrm{OSp}(1|2n)$. The fields $b(X)$ and $f_\alpha(X)$ form a linear supermultiplet (a supersingleton) of a subgroup of $\mathrm{OSp}(1|2n)$ acting linearly in the tensorial superspace. The superfield form of the equations (3.11) and (3.12), both on flat tensorial superspace and on the supergroup manifold $\mathrm{OSp}(1|n)$, have been constructed in [58].

The general solutions of the equations (3.11) and (3.12) are eqs. (3.9) and (3.10) with $\Phi(X,\lambda)$ being the plane wave (3.5). They will prove to be useful for the derivation of the geometrical higher spin equations in $D = 4, 6$ and 10 space-time from the tensorial equations (3.11) and (3.12).



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Let us recall that $X^{\alpha\beta}$ stands for the tensorial coordinates containing the conventional space-time coordinates $x^m$ and the 'helicity' degrees of freedom $y^{m\cdots p}$. In order to make contact with the ordinary space-time picture, one has to single out the $y$-dependence of $b(X)$ and $f_\alpha(X)$ using the decomposition (1.1). For instance, using the form of the general solution (3.5),

$$\Phi(X,\lambda) = \Phi(x,y,\lambda)\, e^{ix^m\,\lambda\gamma^m\lambda}\, e^{iy^{m\cdots p}\,\lambda\gamma_{m\cdots p}\lambda}\,\varphi(\lambda) \tag{3.13}$$

(where the contraction of the spinor indices is implied, e.g. $\lambda\gamma^m\lambda \equiv \lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta$), one finds that in view of (3.9), (3.10) and the Fierz identity

$$(\gamma^m\lambda)_\alpha(\lambda\gamma_m\lambda) = 0 \tag{3.14}$$

for $D = 3, 4, 6$ and 10, the fields $b(x,y)$ and $f_\alpha(x,y)$ satisfy the massless Klein-Gordon equation

$$\partial^m\partial_m b(x^p, y) = 0\,, \tag{3.15}$$

and the Dirac equation

$$\gamma^m\partial_m f(x^p, y) = 0\,. \tag{3.16}$$

We shall now turn to a more detailed analysis of the tensorial equations (3.11) and (3.12) and of their relation to the geometrical equations for the conformal higher spin fields in $D = 3, 4, 6$ and 10 space-time.

## 4. How the quantum dynamics of the tensorial particle produces conformal higher spin fields

### 4.1 $n = 2$, $D = 3$

This case is very simple because there are no extra 'helicity' coordinates: the $y$ variable is absent. This is because a complete basis of symmetric $2\times 2$ matrices is formed by the symmetric $D = 3$ Dirac matrices $\gamma^m_{\alpha\beta}$. Hence,

$$X^{\alpha\beta} = \gamma_m^{\alpha\beta}x^m \quad\Leftrightarrow\quad x^m = \frac{1}{2}\gamma^m_{\alpha\beta}X^{\alpha\beta}\,, \quad (\alpha,\beta = 1,2\,; \quad m = 0,1,2)\,,$$

and $b(X^{\alpha\beta})$ and $f_\alpha(X^{\alpha\beta})$ are simply the $D = 3$ space-time scalar $b(x^m)$ and spinor $f_\alpha(x^m)$ fields.

The only antisymmetric matrix is $\varepsilon_{\alpha\beta}$ playing the role of the charge conjugation matrix. It can be used to lower and raise the spinor indices. Therefore, (3.12) is equivalent to the Dirac equation $\partial_{\alpha\beta}f^\beta(x^m) = (\gamma^m\,\partial_m\,f)_\alpha = 0$, while (3.11) is equivalent to the massless Klein-Gordon equation $\Box\, b(x^m) = 0$. These two equations provide the complete set of $D = 3$ Poincaré group unitary irreps extendable to unitary representations of the $D = 3$ conformal group which, via the isomorphism $\mathrm{Spin}(2,3) \cong \mathrm{Sp}(4,\mathbb{R})$, coincides with the symmetry group $\mathrm{Sp}(4,\mathbb{R})$ of the $n = 2$ tensorial space.

### 4.2 $n=4$, $D=4$

#### 4.2.1 Coordinates

The ten-dimensional tensorial space is parametrized by

$$X^{\alpha\beta} = \frac{1}{2}\, x^m \gamma_m^{\alpha\beta} + \frac{1}{4}\, y^{mn} \gamma_{mn}^{\alpha\beta}\,, \quad (m,n = 0,1,2,3\,; \quad \alpha,\beta = 1,2,3,4)\,, \tag{4.1}$$

where $x^m = 1/2\, X^{\alpha\beta}\gamma^m_{\alpha\beta}$ are associated with the four coordinates of conventional $D=4$ space-time and the six $y^{mn} = 1/2\, X^{\alpha\beta}\gamma^{mn}_{\alpha\beta}$ that describe the spin degrees of freedom. The derivative with respect to $X^{\alpha\beta}$ is

$$\partial_{\alpha\beta} = \frac{1}{2}\, \gamma^m_{\alpha\beta}\, \partial_m + \frac{1}{2}\, \gamma^{mn}_{\alpha\beta}\, \partial_{mn}\,, \tag{4.2}$$

where $\partial_m$ and $\partial_{mn}$ are the derivatives along $x^m$ and $y^{mn}$, respectively.

#### 4.2.2 Unfolded equations

Let us now make a short digression and recall the unfolded formulation of $D=4$ higher-spin fields, which is usually constructed using the two component Weyl spinors (see [4, 8] for details and references). The Majorana spinor index $\alpha$ is then decomposed into a pair of Weyl indices $\alpha = (A, \dot{A})$ with $A, \dot{A} = 1,2$ and $\lambda^\alpha = (\lambda^A\,, \overline{\lambda}^{\dot{A}})$ with $(\lambda^A)^* = \overline{\lambda}^{\dot{A}}$. The momentum constraint (3.3) takes the form

$$P_{AB} = \lambda_A \lambda_B\,, \quad \overline{P}_{\dot{A}\dot{B}} = \overline{\lambda}_{\dot{A}} \overline{\lambda}_{\dot{B}}\,, \quad P_{A\dot{A}} = \lambda_A \overline{\lambda}_{\dot{A}}\,, \tag{4.3}$$

where the last equation is the Cartan-Penrose representation of the light-like momentum. In the same manner, the preonic equation in the $Y$-representation, eq. (3.7), splits into

$$\left(\sigma^{mn}_{AB} \frac{\partial}{\partial y^{mn}} + i \frac{\partial^2}{\partial Y^A \partial Y^B}\right) C(x,y,Y) = 0,$$
$$\left(\overline{\sigma}^{mn}_{\dot{A}\dot{B}} \frac{\partial}{\partial y^{mn}} - i \frac{\partial^2}{\partial \overline{Y}^{\dot{A}} \partial \overline{Y}^{\dot{B}}}\right) C(x,y,Y) = 0$$

and

$$\left(\sigma^m_{A\dot{A}} \frac{\partial}{\partial x^m} + i \frac{\partial^2}{\partial Y^A \partial \bar{Y}^{\dot{A}}}\right) C(x,y,Y) = 0\,, \tag{4.4}$$

where $\sigma^m_{A\dot{A}}$ are the Pauli matrices and $\sigma^{mn}_{AB} = \sigma^{[m}_{A\dot{A}}\, \sigma^{n]\dot{A}}_{B}$.

Equations (4.4) relate the dependence of $C(x,y,Y)$ on $y^{mn}$ to its dependence on $Y^\alpha$. Using this relation one can regard the wave function $C(x^m, Y^\alpha) := C(X^{\alpha\beta}, Y^\alpha)|_{y^{mn}=0}$ at $y^{mn} = 0$ as the fundamental field and thus arrives at the unfolded formulation of [8] whose basic equation for $C(x^m, Y^\alpha)$ is (4.4).

The consistency of (4.4) implies the integrability conditions

$$\frac{\partial^2}{\partial Y^{[A} \partial x^{B]\dot{B}}}\, C(x^{C\dot{C}}, Y) = 0\,, \ \frac{\partial^2}{\partial \bar{Y}^{[\dot{A}} \partial x^{\dot{B}]B}}\, C(x^{C\dot{C}}, Y) = 0\,. \tag{4.5}$$

The expansion of $C(x^m, Y)$ in terms of $Y^A$ and $\overline{Y}^{\dot{A}}$ is

$$C\left(x^p, Y^A, \overline{Y}^{\dot{A}}\right) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} C_{A_1\dots A_m,\, \dot{B}_1\dots \dot{B}_n}(x^p)\, Y^{A_1}\dots Y^{A_m}\, \overline{Y}^{\dot{B}_1}\dots \overline{Y}^{\dot{B}_n}\,, \tag{4.6}$$

where reality imposes $(C_{A_1\dots A_m,\, \dot{B}_1\dots \dot{B}_n})^* = C_{B_1\dots B_n,\, \dot{A}_1\dots \dot{A}_m}$, and the spin-tensors $C$ are by definition symmetric in the indices $A_i$ and in $\dot{B}_i$. All the components of $C(x^m, Y^A, \overline{Y}^{\dot{A}})$ that depend on *both* $Y^A$ and $\overline{Y}^{\dot{A}}$ are auxiliary fields expressed by (4.4) in terms of space-time derivatives of the dynamical fields contained in the analytic fields $C(x^m, Y^A, 0)$ and $C(x^m, 0, Y^{\dot{A}})$. The only dynamical fields are the self-dual and anti-self-dual components $C_{A_1\dots A_{2s}}(x^m)$ and $C_{\dot{A}_1\dots \dot{A}_{2s}}(x^m)$ of the spin-$s$ field strength. The nontrivial equations on the dynamical fields are [8] the Klein-Gordon equation for the spin zero scalar field $\Box C = 0$ and the massless Bargmann-Wigner equations [16] for spin $s > 0$ field strengths

$$\partial^{B\dot{B}} C_{BA_1\dots A_{2s-1}}(x) = 0\,, \qquad \partial^{B\dot{B}} C_{\dot{B}\dot{A}_1\dots \dot{A}_{2s-1}}(x) = 0\,, \tag{4.7}$$

which follow from (4.5).[12]

The massless $D = 4$ higher spin field equations are known to be conformally invariant. So the $SU(2,2)$ symmetry of this infinite set of massless relativistic equations in $D = 4$ gets extended to the $OSp(1|8,\mathbb{R})$ symmetry that becomes more transparent in the tensorial space [4].

Note that $C_{A_1\dots A_{2s}}(x^m)$ and $C_{\dot{A}_1\dots \dot{A}_{2s}}(x^m)$ are related to the integer spin curvature tensor (2.8) and to the half integer spin curvature (2.36) in $D = 4$ as follows

$$R_{m_1 n_1, \cdots, m_s n_s} = \sigma^{A_1 A_{s+1}}_{m_1\, n_1} \cdots \sigma^{A_s A_{2s}}_{m_s\, n_s}\, C_{A_1 \cdots A_s\, A_{s+1} \cdots A_{2s}} + \text{c.c.} \tag{4.8}$$

$$\mathcal{R}^{A_{2s}}_{m_1 n_1, \cdots, m_{s-\frac{1}{2}} n_{s-\frac{1}{2}}} = \sigma^{A_1 A_{s+\frac{1}{2}}}_{m_1\, n_1} \cdots \sigma^{A_{s-\frac{1}{2}} A_{2s-1}}_{m_{s-\frac{1}{2}}\, n_{s-\frac{1}{2}}}\, C^{A_{2s}}_{A_1 \cdots A_{s-\frac{1}{2}}\, A_{s+\frac{1}{2}} \cdots A_{2s-1}} \tag{4.9}$$

#### 4.2.3 The geometric equations from the scalar and spinor field equations in tensorial space

Alternatively to the unfolded construction of section 4.2.2 where we kept the dependence of the wave function on $x^m$ and $Y^\alpha$ and effectively eliminated its dependence on $y^{mn}$, one can deal with the fields $b(x^l, y^{mn})$ and $f_\alpha(x^l, y^{mn})$ and their field equations (3.11) and (3.12) in tensorial space [4]. This formulation will prove to be more convenient for higher-dimensional generalizations.

Since in $D = 4$ the set $\{C^{\beta\gamma}, \gamma_5^{\beta\gamma}, (\gamma_5\gamma_p)^{\beta\gamma}\}$ (where $C^{\beta\gamma}$ is the charge conjugation matrix and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, $(\gamma_5)^2 = -1$) forms a basis of $4\times 4$ antisymmetric matrices, the equation of motion (3.12) of the tensorial space field $f_\alpha(X)$ is equivalent to the system of

[12]The well known counting of the degrees of freedom is as follows: the symmetric tensor $C_{BA_1\dots A_{2s-1}}$ has $\binom{2s+1}{2s} = 2s+1$ components satisfying $\binom{2s}{2s-1} = 2s$ independent conditions; this leaves in $C_{BA_1\dots A_{2s-1}}$ one independent helicity degree of freedom, as is well known for the massless spin $s$ fields in $D = 4$. The spin $s$ state of opposite helicity is described by $C_{\dot{B}\dot{A}_1\dots \dot{A}_{2s-1}}(x^m)$.

*linearly dependent* differential equations

$$(\gamma^m \partial_m - \gamma^{mn}\partial_{mn}) f = 0\,,$$
$$(\gamma^m \partial_m - \gamma^{mn}\partial_{mn})\gamma_5 f = 0\,,$$
$$(\gamma^m \partial_m - \gamma^{mn}\partial_{mn})\gamma_5\gamma_p\, f = 0\,, \tag{4.10}$$

where the expression (4.2) for the tensorial partial derivative has been used.

Moving $\gamma_5$ and $\gamma_5\gamma_p$ to the left hand side of (4.10) and taking linear combinations of the resulting equations, one gets the following equivalent set of *independent* equations

$$\gamma^p\, \partial_p\, f(x^l, y^{mn}) = 0\,, \tag{4.11}$$
$$(\partial_p - 2\gamma^r\, \partial_{rp})\, f(x^l, y^{mn}) = 0\,. \tag{4.12}$$

From eqs. (4.11) and (4.12) one can derive the equation

$$\partial_{mn}\, f = \frac{1}{2}\,(\partial_{mn} + \frac{1}{2}\epsilon_{mnpq}\, \partial^{pq}\, \gamma_5)\, f + \frac{1}{2}\, \gamma_{[m}\, \partial_{n]}\, f\,, \tag{4.13}$$

which describes the decomposition of the spinor-tensor $\partial_{mn}\, f$ into the self-dual gamma-traceless part $(\partial_{mn} + \frac{1}{2}\epsilon_{mnpq}\, \partial^{pq}\, \gamma_5)\, f$

$$\gamma^m \left(\partial_{mn} + \frac{1}{2}\epsilon_{mnpq}\, \partial^{pq}\, \gamma_5\right) f = 0\,,$$
$$\left(\partial_{mn} + \frac{1}{2}\epsilon_{mnpq}\, \partial^{pq}\, \gamma_5\right) f = \frac{1}{2}\, \epsilon_{mnrs}\, \gamma_5 \left(\partial^{rs} + \frac{1}{2}\epsilon^{rspq}\, \partial_{pq}\, \gamma_5\right) f \tag{4.14}$$

and the 'tracefull' part which is proportional to the $D=4$ space-time derivative of $f(x,y)$, i.e. $\frac{1}{2}\, \gamma_{[m}\, \partial_{n]}\, f$.

Analogously to the fermionic equations, the equation of motion (3.11) of the tensorial space scalar $b(x,y)$ is equivalent to

$$\partial_p\, \partial^p\, b(x^l, y^{mn}) = 0\,, \tag{4.15}$$
$$\left(\partial_p\, \partial_q - 4\, \partial_{pr}\, \partial^r{}_q\right)\, b(x^l, y^{mn}) = 0\,, \tag{4.16}$$
$$\epsilon^{pqrt}\partial_{pq}\, \partial_{rs}\, b(x^l, y^{mn}) = 0\,, \tag{4.17}$$
$$\epsilon^{pqrt}\partial_q\, \partial_{rt}\, b(x^l, y^{mn}) = 0\,, \tag{4.18}$$
$$\partial_q{}^p\, \partial_p\, b(x^l, y^{mn}) = 0\,. \tag{4.19}$$

Roughly speaking, the system of equations (4.15)–(4.19), which also holds for the spinor field, is the "square" of eqs. (4.11)–(4.12), because the former can be obtained from the latter as integrability conditions and using the duality relation

$$\gamma_{mn} = \frac{1}{2}\, \epsilon_{mnpq}\, \gamma_5 \gamma^{pq}\,. \tag{4.20}$$

For further generalization to higher dimensions $D = 6$ and $D = 10$ it is instructive to derive equations (4.11)–(4.19) by applying the derivatives $\partial_m$ and $\partial_{mn}$ to the general solutions (3.9) and (3.10) of the tensorial equations (3.11) and (3.12) and using $\gamma$-matrix Fierz identities.

This way of deriving the Dirac (4.11) and Klein-Gordon (4.15) equations has already been explained at the end of Sec. 3, so we proceed with the consideration of the other equations.

To get (4.12) we take the derivative $\gamma^r \partial_{rp}$ of (3.10), where $\Phi(X, \lambda)$ is the plane wave (3.13), and notice that

$$2(\lambda\gamma^r)^\alpha (\lambda\gamma_{rp}\lambda) = \lambda^\alpha (\lambda\gamma_p\lambda) \tag{4.21}$$

holds due to the well known Fierz identity in $D = 3, 4, 6$ and 10

$$\gamma_{m\,(\alpha\beta}\,\gamma^m_{\gamma)\delta} = 0\,. \tag{4.22}$$

Eq. (4.16) is obtained by taking and comparing the second derivatives of (3.9) and (3.13), and noticing that, as a consequence of eq. (4.21),

$$4\,(\lambda\gamma^{mp}\lambda)\,(\lambda\gamma_p{}^n\lambda) = (\lambda\gamma^m\lambda)\,(\lambda\gamma^n\lambda)\,. \tag{4.23}$$

In the same way one checks that eqs. (4.17), (4.18) and (4.19) hold, respectively, due to the algebraic identities

$$\epsilon_{mnpq}\,(\lambda\gamma^{mn}\lambda)\,(\lambda\gamma^{pq}\lambda) = 2\,(\lambda\gamma_5\gamma^{mn}\lambda)\,(\lambda\gamma_{mn}\lambda) = 0\,, \tag{4.24}$$

$$\epsilon_{mnpq}\,(\lambda\gamma^{mn}\lambda)\,(\lambda\gamma^p\lambda) = 2\,(\lambda\gamma_5\gamma^{mn}\lambda)\,(\lambda\gamma_n\lambda) = 0\,, \tag{4.25}$$

$$(\lambda\gamma^m\lambda)\,(\lambda\gamma_{mn}\lambda) = 0\,. \tag{4.26}$$

Note that all the identities (4.23)–(4.26) are consequences of (4.21). This explains from the twistor-like point of view why the set of eqs. (4.11) and (4.12) is the "square root" of (4.15)–(4.19).

Let us now analyze the physical meaning of the equations (4.11)–(4.19) from the point of view of the effective four-dimensional field theory. As we shall show, eqs. (4.17), (4.18) and (4.19) produce, respectively, the first (eq. (2.9)) and the second (eq. (2.10)) Bianchi identities, and the transversality condition (2.11) for the $D = 4$ higher spin curvatures.

To this end let us expand $b(x, y)$ and $f_\alpha(x, y)$ in power series[13] of $y^{mn}$

$$\begin{aligned} b(x,\,y) \;=\; & \phi(x) + y^{m_1n_1}F_{m_1n_1}(x) + y^{m_1n_1}\,y^{m_2n_2}\,\hat{R}_{m_1n_1,m_2n_2}(x) + \\ & + \sum_{s=3}^{\infty} y^{m_1n_1}\cdots y^{m_sn_s}\,\hat{R}_{m_1n_1,\cdots,m_sn_s}(x)\,, \end{aligned} \tag{4.27}$$

$$f^\alpha(x,y) = \psi^\alpha(x) + y^{m_1n_1}\,\hat{\mathcal{R}}^\alpha_{m_1n_1}(x) + \sum_{s=\frac{5}{2}}^{\infty} y^{m_1n_1}\cdots y^{m_{s-\frac12}n_{s-\frac12}}\,\hat{\mathcal{R}}^\alpha_{m_1n_1,\cdots,m_{s-\frac12}n_{s-\frac12}}(x)\,. \tag{4.28}$$

[13]Such an expansion is justified by the presence in the general solutions (3.9) and (3.10) of the tensorial equations (3.11) and (3.12) of the plane wave function (3.13) which allows us to expand $e^{i\lambda\gamma_{mn}\lambda\, y^{mn}}$ in power series.

In the multi index notation of section 2 these series take the form

$$b(x,\,y) \;=\; \phi(x) + y^{[2]} F_{[2]}(x) + y^{[2]_1}\, y^{[2]_2}\, \hat{R}_{[2]_1[2]_2}(x) + \sum_{s=3}^{\infty} y^{[2]_1} \cdots y^{[2]_s}\, \hat{R}_{[2]_1\cdots\,[2]_s}(x)\,, \quad (4.29)$$

$$f^{\alpha}(x,y) \;=\; \psi^{\alpha}(x) + y^{[2]}\, \hat{\mathcal{R}}^{\alpha}_{[2]}(x) + \sum_{s=\frac{5}{2}}^{\infty} y^{[2]_1} \cdots y^{[2]_{s-\frac{1}{2}}}\, \hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x)\,. \quad (4.30)$$

The scalar field $\phi(x)$ and the spinor field $\psi^{\alpha}(x)$ as well as all the higher order tensors and spin tensors in (4.27) and (4.28) satisfy the Klein-Gordon equation (4.15) and hence are massless $D=4$ fields. The fermionic fields $\psi^{\alpha}(x)$ and $\hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x)$ satisfy the Dirac equation (4.11).

Eq. (4.12) tells us that the gamma-trace of the fermionic spin $s$ tensor $\hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x)$ is proportional to the space-time derivative of the spin $(s-1)$ tensor $\hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{3}{2}}}(x)$, that means for instance

$$(\gamma^{m_1})^{\alpha}{}_{\beta}\hat{\mathcal{R}}^{\beta}_{m_1n_1} \;=\; \frac{1}{2}\partial_{n_1}\psi^{\alpha}\,,$$
$$(\gamma^{m_1})^{\alpha}{}_{\beta}\hat{\mathcal{R}}^{\beta}_{m_1n_1,m_2n_2} \;=\; \frac{1}{4}\partial_{n_1}\hat{\mathcal{R}}^{\alpha}_{m_2n_2}\,,$$

etc. As a consequence, the fermionic spin $s$ tensor $\hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x)$ decomposes as follows

$$\hat{\mathcal{R}}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x) = \mathcal{R}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x) + \sum_{k=1}^{s-\frac{1}{2}} a_k\, \partial_{[m_1}\,\gamma_{n_1]}\ldots\partial_{[m_k}\,\gamma_{n_k]}\hat{\mathcal{R}}_{[2]_{k+1}\cdots\,[2]_{s-\frac{1}{2}}}(x)\,, \quad (4.31)$$

where according to our notation (eqs. (2.2) and (2.4)) all the pairs of the antisymmetric indices $[m_i\, n_i]$ are symmetrized (with weight one), and $\mathcal{R}^{\alpha}_{[2]_1\cdots\,[2]_{s-\frac{1}{2}}}(x)$ is gamma-traceless as in eq. (2.41). The coefficients $a_k$ can be determined iteratively using eq. (4.12): $a_{2k-1} = 2ka_{2k}$, $a_{2k} = 2(s-k+1/2)a_{2k+1}$, $a_1 = -\frac{1}{2(s-1/2)}$. Therefore the gamma-trace parts of the higher rank spin-tensors do not describe any independent physical higher spin degrees of freedom.

Eq. (4.16) implies that starting with spin 2 the trace of the bosonic spin $s$ tensor is proportional to the second space-time derivative of the spin $s-2$ tensor, *e.g.*

$$\hat{R}_{m_1n_1,}{}^{n_1}{}_{n_2} = \frac{1}{8}\partial_{m_1}\partial_{n_2}\phi\,, \qquad \hat{R}_{m_1n_1,}{}^{n_1}{}_{n_2,m_3n_3} = \frac{1}{24}\partial_{m_1}\partial_{n_2}F_{m_3n_3}$$

and so on for all higher spins. Analogously to the case of the half-integer higher spin fields one can extract the traceless part of the curvature decomposing $\hat{R}_{[2]_1\cdots\,[2]_s}$ in the following way

$$\hat{R}_{[2]_1\cdots\,[2]_s}(x) \;=\; R_{[2]_1\cdots\,[2]_s}(x) - \frac{1}{2s}\partial_{[m_1}\eta_{n_1][n_2}\partial_{m_2]}\hat{R}_{[2]_3\cdots\,[2]_s}(x) +$$
$$+ \sum_{k=2}^{[\frac{s}{2}]} b_k\partial_{[m_1}\eta_{n_1][n_2}\partial_{m_2]}\ldots\partial_{[m_{2k-1}}\eta_{n_{2k-1}][n_{2k}}\partial_{m_{2k}]}\hat{R}_{[2]_{2k+1}\cdots\,[2]_s}(x)\,, \quad (4.32)$$

where $R_{[2]_1\cdots[2]_s}(x)$ is traceless as in eq. (2.19), and all the pairs of the antisymmetric indices are symmetrized (with unit weight). The exact values of the coefficients $b_k$, which can be determined iteratively as in the case of the half integer higher spins, are not important for further analysis. The structure of (4.32) tells us that the traces of the higher rank tensors do not describe independent higher spin degrees of freedom.

Eqs. (4.17), (4.18) and (4.19) require that the tensor fields $\hat{R}_{[2]_1\cdots[2]_s}(x)$ and $\hat{\mathcal{R}}^{\alpha}_{[2]_1\cdots[2]_{s-\frac{1}{2}}}(x)$ as well as the (gamma-) traceless fields $R_{[2]_1\cdots[2]_s}(x)$ and $\mathcal{R}^{\alpha}_{[2]_1\cdots[2]_{s-\frac{1}{2}}}(x)$ satisfy the Bianchi identities, eqs. (2.9) and (2.10), and that they are co-closed (2.11). Thus, in accordance with the general discussion of section 2, the traceless $R_{[2]_1\cdots[2]_s}(x)$ and gamma-traceless $\mathcal{R}^{\alpha}_{[2]_1\cdots[2]_{s-\frac{1}{2}}}(x)$ tensor fields are the curvatures of the higher spin gauge field potentials satisfying 'geometric' equations of motion. For instance $F_{mn}$ is the on-shell Maxwell field strength and $R_{m_1n_1,m_2n_2}$ is the linearized on-shell Riemann curvature.

We have thus reviewed how the free geometric equations for the infinite set of higher spin field strengths in $D=4$ space-time arise from the simple scalar (3.11) and spinor (3.12) field equations in $n=4$ tensorial space. The $\mathrm{OSp}(1|2n)=\mathrm{OSp}(1|8)$ invariance of the tensorial equations implies the $\mathrm{OSp}(1|8)$ generalized superconformal invariance of the infinite system of the geometric integer and half integer spin equations. Each physical field of spin $s$ appears in this infinite spectrum only once.

### 4.3 $n=16$, $D=10$

We now turn to the more complicated case of the derivation of the conformal higher spin geometrical equations in 10-dimensional space from the $n=16$ tensorial space equations (3.11) and (3.12).

#### 4.3.1 Coordinates

In this case, the twistor-like variable $\lambda_\alpha$ is a 16-component Majorana-Weyl spinor. The gamma-matrices $\gamma_m^{\alpha\beta}$ and $\gamma_{m_1\cdots m_5}^{\alpha\beta}$ form a basis of the symmetric $16\times16$ matrices, so the $n=16$ tensorial manifold is parametrized by the coordinates

$$X^{\alpha\beta}=\frac{1}{16}\left(x^m\gamma_m^{\alpha\beta}+\frac{1}{2\cdot 5!}\,y^{m_1\ldots m_5}\gamma^{\alpha\beta}_{m_1\ldots m_5}\right)=X^{\beta\alpha}\,,$$
$$(m=0,1,\ldots,9;\quad \alpha,\beta=1,2,\ldots,16)\,, \tag{4.33}$$

where

$$x^m=X^{\alpha\beta}\gamma^m_{\alpha\beta}$$

are associated with the coordinates of the $D=10$ space-time, while the anti-self-dual coordinates

$$y^{m_1\ldots m_5}=X^{\alpha\beta}\gamma_{\alpha\beta}^{m_1\ldots m_5}=-\frac{1}{5!}\,\epsilon^{m_1\ldots m_5 n_1\ldots n_5}y_{n_1\ldots n_5}\,,$$

describe spin degrees of freedom. The derivative with respect to $X^{\alpha\beta}$ is therefore given by

$$\partial_{\alpha\beta}=\gamma^m_{\alpha\beta}\,\partial_m+\gamma_{\alpha\beta}^{m_1\ldots m_5}\,\partial_{m_1\ldots m_5}=\gamma^m_{\alpha\beta}\,\partial_m+\gamma^{[5]}_{\alpha\beta}\,\partial_{[5]}\,, \tag{4.34}$$

where $\partial_{m_1\ldots m_5} \equiv \partial_{[5]}$ is the derivative with respect to $y^{m_1\ldots m_5} \equiv y^{[5]}$, which because of self-duality has the following property

$$\partial_{m_1\ldots m_5}\, y^{n_1\ldots n_5} = \frac{1}{2}\left(\delta^{n_1\cdots n_5}_{m_1\cdots m_5} + \frac{1}{5!}\,\epsilon_{m_1\ldots m_5}{}^{n_1\ldots n_5}\right), \tag{4.35}$$

where $\delta^{n_1\cdots n_i}_{m_1\cdots m_i} \equiv \delta^{[n_1}_{m_1}\,\delta^{n_2}_{m_2}\cdots\,\delta^{n_i]}_{m_i}$.

### 4.3.2 Field equations

The matrices $\gamma^{\beta\gamma}_{m_1m_2m_3}$ form a basis of the antisymmetric $16\times 16$ matrices, therefore the $n=16$ spinor equation (3.12) is equivalent to the equation

$$(\gamma^n\partial_n + \gamma^{n_1\ldots n_5}\partial_{n_1\ldots n_5})\gamma_{m_1m_2m_3} f(x,y) = 0\,. \tag{4.36}$$

By virtue of the gamma matrix properties (A.5), the multiplication of (4.36) by $\gamma^{m_1m_2m_3}$ leads to the Dirac equation

$$\gamma^m\partial_m f(x,y) = 0\,. \tag{4.37}$$

Now taking into account (4.37), using the identity (A.4) of the appendix and the duality relations between $\gamma$-matrices we can rewrite eq. (4.36) in a simpler form

$$\left(6\,\gamma_{[m_1m_2}\,\partial_{m_3]} - 5!\,\gamma^{n_1n_2}\partial_{m_1m_2m_3n_1n_2} - 5!\,\gamma^{n_1n_2n_3n_4}\,\eta_{n_4[m_1}\,\partial_{m_2m_3]n_1n_2n_3}\right) f(x,y) = 0\,. \tag{4.38}$$

Multiplying eq. (4.38) by $\gamma_{m_4}$ and anti-symmetrizing the indices we get

$$\Big(-6\,\gamma_{[m_1m_2m_3}\partial_{m_4]} - 2\cdot 5!\,\gamma^{m_5}\,\partial_{m_1m_2m_3m_4m_5} - \\ - 2\cdot 5!\,\gamma^{n_1n_2}{}_{[m_1}\,\partial_{m_2m_3m_4]n_1n_2} - 5!\,\gamma^{n_1n_2n_3}{}_{[m_1m_2}\partial_{m_3m_4]n_1n_2n_3}\Big) f(x,y) = 0\,. \tag{4.39}$$

We now notice that because of the self-duality of $\gamma^{[5]\alpha\beta}$ and $\partial_{[5]}$ the last term in (4.39) is identically zero

$$-5!\,\gamma^{n_1n_2n_3}{}_{[m_1m_2}\partial_{m_3m_4]n_1n_2n_3}\, f(x,y) \equiv 0\,,$$

so (4.39) reduces to

$$\left(-6\,\gamma_{[m_1m_2m_3}\partial_{m_4]} - 2\cdot 5!\,\gamma^{m_5}\partial_{m_1m_2m_3m_4m_5} - 2\cdot 5!\,\gamma^{n_1n_2}{}_{[m_1}\,\partial_{m_2m_3m_4]n_1n_2}\right) f(x,y) = 0\,. \tag{4.40}$$

As is explained below, this equation splits into

$$(2\,\gamma_{[m_1m_2m_3}\partial_{m_4]} - 5!\,\gamma^{m_5}\partial_{m_1m_2m_3m_4m_5}) f(x,y) = 0\,, \tag{4.41}$$

which is an analogue of (4.12), plus

$$\left(5\,\gamma_{[m_1m_2m_3}\partial_{m_4]} + 5!\,\gamma^{n_1n_2}{}_{[m_1}\,\partial_{m_2m_3m_4]n_1n_2}\right)\, f(x,y) = 0\,. \tag{4.42}$$

Then, as a consequence of (4.41) and (4.38)

$$\left(5\gamma^{n_1n_2}\partial_{m_1m_2m_3n_1n_2} + 3\,\gamma^{n_1n_2n_3n_4}\,\eta_{n_4[m_1}\,\partial_{m_2m_3]n_1n_2n_3}\right) f(x,y) = 0\,. \tag{4.43}$$

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A simple way to arrive at (4.41) and (4.42) is to consider the general twistor-like solution (3.10), (3.13) of the tensorial fermionic equation (3.12). Acting on

$$f_\beta(X) = -i \int d\lambda^{16}\, e^{i\frac{1}{32\times 5!}(\lambda\gamma_{m_1\ldots m_5}\lambda)\,y^{m_1\ldots m_5}}\; e^{i\frac{1}{16}(\lambda\gamma_m\lambda)\,x^m}\,\lambda_\beta\,\varphi(\lambda)\; + \text{c.c.} \tag{4.44}$$

with $5!\,\gamma^{p_5\,\alpha\beta}\partial_{p_1p_2p_3p_4p_5}$ we get (up to the factor $\frac{1}{16}$) the following trilinear combination of $\lambda$'s

$$\frac{1}{2}\,(\lambda\gamma_{m_1\ldots m_5}\lambda)(\gamma^{m_5}\lambda)^\alpha.$$

Then, using the identities (A.3), (A.3) of the appendix, the basic cyclic identity (4.22) and taking into account that $\lambda\gamma_{m_1m_2m_3}\lambda \equiv 0$ we find that

$$\frac{1}{2}(\lambda\gamma_{m_1\ldots m_4m_5}\lambda)(\gamma^{m_5}\lambda)^\alpha = \frac{1}{2}(\lambda\gamma_{m_1\ldots m_4}\,\gamma_{m_5}\lambda)\,(\gamma^{m_5}\lambda)^\alpha \tag{4.45}$$

$$= -\frac{1}{4}(\gamma_{m_5}\,\gamma_{m_1\ldots m_4}\lambda)^\alpha\,(\lambda\gamma^{m_5}\lambda) \tag{4.46}$$

$$= 2(\,\gamma_{[m_1m_2m_3}\lambda)^\alpha\,(\lambda\gamma_{m_4]}\lambda)\,. \tag{4.47}$$

On the other hand, expression (4.47) is obtained (up to the factor $\frac{1}{16}$) by the action of $2\,\gamma^{\alpha\beta}_{[m_1m_2m_3}\partial_{m_4]}$ on (4.44). This completes the twistor-like proof of (4.41).

To prove (4.42) we multiply (4.45) and (4.47) by $\gamma_{n_4}{}^{m_4}$, antisymmetrize the indices $[m_1m_2m_3n_4]$ and use the relations (A.3), (A.3), (4.45) and (4.47) to get

$$\frac{1}{2}\,(\lambda\gamma_{m_4m_5[m_1m_2m_3}\lambda)(\gamma_{n_4]}{}^{m_4m_5}\lambda)^\alpha = 5\,(\,\gamma_{[m_1m_2m_3}\lambda)^\alpha\,(\lambda\gamma_{n_4]}\lambda)\,, \tag{4.48}$$

which is the algebraic twistor-like solution of the eq. (4.42).

The above twistor-like analysis implies that among the equations (4.38)–(4.42) and (4.43) only one is independent, while the others follow from it, provided the Dirac equation (4.37) holds (which can also be checked directly). For instance, we can consider eqs. (4.37) and (4.12) as the independent fermionic equations which replace (4.36)

$$\gamma^m\partial_m f(x,y) = 0\,, \tag{4.49}$$

$$(2\,\gamma_{[m_1m_2m_3}\partial_{m_4]} - 5!\,\gamma^{m_5}\partial_{m_1m_2m_3m_4m_5})f(x,y) = 0\,. \tag{4.50}$$

Analogously, the tensorial equation (3.11) for the field $b(x)$, which in $D=10$ is equivalent to

$$tr[(\gamma^m\partial_m + \gamma^{m_1\ldots m_5}\partial_{m_1\ldots m_5})\gamma_{p_1p_2p_3}(\gamma^n\partial_n + \gamma^{n_1\ldots n_5}\partial_{n_1\ldots n_5})\gamma_{q_1q_2q_3}]b(x,y) = 0\,, \tag{4.51}$$

reduces to the following set of equations

$$\partial^p\partial_p\, b(x,y) = 0\,, \tag{4.52}$$

$$\Big(\delta^{[n_1n_2n_3}_{[m_1\,m_2\,m_3}\partial^{n_4]}\partial_{m_4]} - 5!\,\delta^{[n_1}_{[m_1}\,\partial_{m_2}\partial_{m_3m_4]}{}^{n_2n_3n_4]} + 5\cdot 5!\,\partial^{n_1n_2n_3n_4p}\,\partial_{m_1m_2m_3m_4p}\Big)\,b(x,y) = 0\,, \tag{4.53}$$

$$\partial^{m_5}\partial_{m_1\ldots m_5}\, b(x,y) = 0\,. \tag{4.54}$$

Because of the self-duality of $\partial_{m_1\ldots m_5}\, b(x,y)$ the Bianchi identity

$$\partial_{[n}\,\partial_{m_1\ldots m_4 m_5]}\, b(x,y)=0 \tag{4.55}$$

follows from the transversality condition (4.54).

As in the case of the fermionic equations, a simple way to derive eqs. (4.52)–(4.54) is to make use of the general solution (3.9) of eq. (3.11), which, in turn, is a consequence of the general plane wave solution (3.13) of the $n=16$ preonic equation (3.4)

$$b(X)=\int d\lambda^{16}\, e^{i\frac{1}{32\times 5!}(\lambda\gamma_{m_1\ldots m_5}\lambda)\, y^{m_1\ldots m_5}}\, e^{i\frac{1}{16}(\lambda\gamma_m\lambda)\, x^m}\, \varphi(\lambda)\, +\text{c.c.}\,. \tag{4.56}$$

The Klein-Gordon equation (4.52) for $b(x,y)$ has already been derived in this fashion (see eqs. (3.14) and (3.15)).

To check (4.54) we take the second derivative $\partial^{m_5}\partial_{m_1\ldots m_5}$ of (4.56) and observe that it is indeed zero because of the identity

$$(\lambda\gamma_{m_1\ldots m_4 m_5}\lambda)(\lambda\gamma^{m_5}\lambda)=0\,, \tag{4.57}$$

which in view of $\lambda\,\gamma_{m_1m_2m_3}\,\lambda\equiv 0$ follows from (4.45)–(4.47).

To check (4.53) we multiply (4.45) and (4.47) by $\frac{1}{2\cdot 4!}\,(\lambda\gamma^{n_1\cdots\, n_4})_\alpha$ and use the identity (A.4) to get

$$\begin{aligned}&\frac{1}{4\cdot 4!}\,(\lambda\gamma^{n_1\cdots\, n_4 p}\lambda)\,(\lambda\gamma_{m_1\cdots\, m_4 p}\lambda)=\\ &\quad=\frac{1}{2}\,\delta^{[n_1}_{[m_1}\,(\lambda\gamma_{m_2}\lambda)\,(\lambda\gamma^{n_2n_3n_4]}{}_{m_3m_4]}\lambda)-\,\delta^{[n_1n_2n_3}_{[m_1m_2m_3}\,(\lambda\gamma_{m_4]}\lambda)\,(\lambda\gamma^{n_4]}\lambda)\,,\end{aligned}$$

which is the twistor-like analog of (4.53).

Let us now show that the equations (4.50)–(4.55) comprise the system of the geometrical equations for the field strengths of the conformal higher spin fields in ten-dimensional space-time. The expansions of $b(x,\,y)$ and $f_\alpha(x,y)$ in series of $y^{[5]}\equiv y^{m_1\cdots m_5}$ are

$$\begin{aligned}b(x,\,y)\,&=\,\phi(x)+y^{[5]}F_{[5]}(x)+y^{[5]_1}\,y^{[5]_2}\,\hat{R}_{[5]_1[5]_2}(x)+\sum_{s=3}^{\infty}\,y^{[5]_1}\cdots y^{[5]_s}\,\hat{R}_{[5]_1\,\cdots\,[5]_s}(x)\,,\\ f_\alpha(x,\,y)\,&=\,\psi_\alpha(x)+y^{[5]}\hat{\mathcal{R}}_{\alpha\,[5]}(x)+\sum_{s=5/2}^{\infty}\,y^{[5]_1}\cdots y^{[5]_{s-1/2}}\,\hat{\mathcal{R}}_{\alpha\,[5]_1\cdots\,[5]_{s-1/2}}(x)\,.\end{aligned} \tag{4.58}$$

The scalar field $\phi(x)$ and the spinor field $\psi_\alpha(x)$, as well as all the higher order tensors and spinor-tensors in (4.58), satisfy the Klein Gordon equation (4.52) and hence are massless $D=10$ fields. The fermionic fields $\psi_\alpha(x)$ and $\hat{\mathcal{R}}_{\alpha\,[5]_1\cdots\,[5]_{s-\frac{1}{2}}}(x)$ satisfy the Dirac equation (4.49).

As in $D=4$, eq. (4.41) relates the gamma-trace of the rank $5\times s-\frac{1}{2}$ tensor to the first derivative of the rank $5\times(s-1)$ tensor, e.g.

$$\gamma^{r_1}\hat{\mathcal{R}}_{m_1n_1p_1q_1r_1}=\frac{2}{5!}\gamma_{[m_1n_1p_1}\partial_{q_1]}\psi\,.$$

Therefore in complete analogy with $D = 4$ case on can extract from the spinor-tensor $\hat{\mathcal{R}}_{\alpha\,[5]_1\cdots[5]_{s-\frac{1}{2}}}$ its gamma traceless part with the help of eq. (4.50) which implies that

$$\hat{\mathcal{R}}_{\alpha\,[5]_1\cdots[5]_{s-\frac{1}{2}}} = \mathcal{R}_{\alpha\,[5]_1\cdots[5]_{s-\frac{1}{2}}} + \frac{1}{2\cdot 5!(s-\frac{1}{2})}\,\partial_{[m_1}\,(\gamma_{n_1p_1q_1r_1]}\,\hat{\mathcal{R}})_{\alpha\,[5]_2\cdots[5]_{s-\frac{1}{2}}} + \cdots\,, \tag{4.59}$$

where, according to our notation, the groups of five antisymmetric indices are symmetrized (with weight one), and $\mathcal{R}_{\alpha\,[5]_1\cdots[5]_{s-\frac{1}{2}}}$ is $\gamma$-traceless as in eq. (2.41). The dots stand for terms proportional to higher order derivatives of lower rank spinor-tensors.

A novelty of the $D = 10$ case with respect to $D = 4$ is that eq. (4.53) relates the trace of the rank $5\times s$ tensor not only to the second derivative of the rank $5\times(s-2)$ tensor but also to the first derivative of the rank $5\times(s-1)$ tensor, e.g.

$$\hat{R}^{m_1n_1p_1q_1r_1,}{}_{m_2n_2p_2q_2r_1} = -\frac{1}{10\cdot 5!}\partial^{[m_1}\delta^{n_1}_{[m_2}\delta^{p_1}_{n_2}\delta^{q_1]}_{p_2}\partial_{q_2]}\phi + \frac{1}{10}\partial^{[m_1}\delta^{n_1}_{[m_2}F^{p_1q_1]}{}_{n_2p_2q_2]}\,. \tag{4.60}$$

The traceless part $R_{[5]_1\cdots[5]_s}$ of the tensor $\hat{R}_{[5]_1\cdots[5]_s}$ can be extracted with the help of eq. (4.53).

The traceless rank $5s$ tensor $R_{[5]_1\cdots[5]_s}$ is automatically irreducible under $\mathrm{GL}(10,\mathbb{R})$ due to the self-duality property, and it is thus associated with the rectangular Young diagram $(s,s,s,s,s)$ which is made of five rows of equal length $s$. Eq. (4.54) is the transversality condition, hence the rank $5s$ tensor is harmonic (again due to self-duality) and satisfies the Bianchi identities (4.55). In accordance with the general considerations of section 2, this implies that the traceless tensor $R_{[5]_1\cdots[5]_s}$ is indeed the field strength of a chiral spin $s$ gauge field $\phi_{[4]_1\cdots[4]_s}$ and that the gamma-traceless spinor-tensor $\mathcal{R}_{\alpha\,[5]_1\cdots[5]_{s-\frac{1}{2}}}$ is the field strength of a fermionic chiral spin $s$ gauge field $\psi_{\alpha\,[4]_1\cdots[4]_{s-\frac{1}{2}}}$, whose symmetry properties are described by the rectangular Young diagram $(s,s,s,s)$ [34].

To summarize, the physical states of the quantum $n = 16$ tensorial particle form a representation of $\mathrm{OSp}(1|32,\mathbb{R})$ which in $D = 10$ decomposes into an infinite sum containing all the chiral integer and half-integer higher spin representations of the conformal group $Spin(2,10) \subset \mathrm{OSp}(1|32,\mathbb{R})$ associated with space-time fields that satisfy the proper geometrical field equations. Each physical field of spin $s$ appears in the spectrum only once.

### 4.4 $n = 8$, $D = 6$

#### 4.4.1 Coordinates

The commuting spinor $\lambda_\alpha$ is now a symplectic Majorana-Weyl spinor (see e.g. [55] for details). The spinor index can thus be decomposed as $\alpha = a\otimes i$ ($\alpha = 1,\ldots,8$; $a = 1,2,3,4$; $i = 1,2$). The tensorial space coordinates $X^{\alpha\beta} = X^{ai\,bj}$ are decomposed into

$$\begin{aligned} X^{ai\,bj} &= \frac{1}{8}\,x^m\,\tilde{\gamma}^{ab}_m\,\epsilon^{ij} + \frac{1}{16\cdot 3!}\,y^{mnp}_I\,\tilde{\gamma}^{ab}_{mnp}\,\tau^{ij}_I\,,\\ m,n,p &= 0,\ldots,5\,;\quad a,b = 1,\ldots,4\,;\quad i,j = 1,2\,;\quad I = 1,2,3\,, \end{aligned} \tag{4.61}$$

where $\epsilon^{12} = -\epsilon_{12} = 1$, and $\tau^{ij}_I$ ($I = 1,2,3$) provide a basis of $2\times 2$ symmetric matrices and are expressed through the usual $\mathrm{SU}(2)$ group Pauli matrices, $\tau_{I\,ij} = \epsilon_{jj'}\,\sigma_{I\,i}{}^{j'}$; below we also use $\tau^{ij}_I = \epsilon^{ii'}\,\sigma_{I\,i'}{}^j$ (see appendix for further details). The matrices $\tilde{\gamma}^{ab}_m$

($\gamma^m_{ab} = 1/2\,\varepsilon_{abcd}\tilde{\gamma}^{m\,cd}$) provide a complete set of $4\times 4$ antisymmetric matrices with upper [lower] indices transforming under an [anti]chiral fundamental representation of the non-compact group $SU^*(4) \sim Spin(1,5)$. For the space of $4\times 4$ symmetric matrices with upper [lower] indices a basis is provided by the set of self-dual [anti-self-dual] matrices $(\tilde{\gamma}^{mnp})^{ab}$ $[\gamma^{mnp}_{ab}]$,

$$(\tilde{\gamma}^{mnp})^{ab} = \frac{1}{3!}\epsilon^{mnpqrs}\tilde{\gamma}^{ab}_{qrs}\,, \qquad \gamma_{ab}^{mnp} = -\frac{1}{3!}\epsilon^{mnpqrs}(\gamma_{qrs})_{ab}\,. \tag{4.62}$$

The coordinates

$$x^m \,=\, x^{ai\,bj}\,\gamma^m_{ab}\,\epsilon_{ij}\,, \tag{4.63}$$

are associated with $D=6$ space-time, while the self-dual coordinates

$$y^{mnp}_I = x^{ai\,bj}\,\gamma^{mnp}_{ab}\,\tau_{I\,ij} = -\frac{1}{3!}\,\epsilon^{mnpqrs}y^I_{qrs}\,, \tag{4.64}$$

describe spinning degrees of freedom. The coefficients in (4.63), (4.64) are chosen in such a way that the derivative with respect to $X^{\alpha\beta}$ is decomposed on the vector derivative $\partial_m$ and the self-dual tensorial derivative $\partial^I_{mnp} = \frac{\partial}{\partial y^{mnp}_I}$ with the unity coefficients,

$$\partial_{ai\,bj} \,=\, \gamma^m_{ab}\,\epsilon_{ij}\,\partial_m \,+\, \gamma^{mnp}_{ab}\,\tau^I_{ij}\,\partial^I_{mnp}\,. \tag{4.65}$$

The self-duality of

$$\partial^I_{mnp} = \frac{1}{3!}\epsilon_{mnpqrs}\partial^{I\,qrs} \tag{4.66}$$

implies that

$$\tilde{\gamma}^{mnp}\,\partial^I_{mnp} = 0\,, \qquad \tilde{\gamma}_{[m_1}{}^{np}\,\partial^I_{m_2]np} = 0\,, \qquad \tilde{\gamma}_{[m_1m_2}{}^{n}\,\partial^I_{m_2m_4]n} = 0\,. \tag{4.67}$$

Note that in eq. (4.65) $\gamma^{mnp}_{ab}\,\partial^I_{mnp} \neq 0$, because $\gamma^{mnp}_{ab}$ is anti-self-dual and $\partial^I_{mnp}$ is self-dual. Let us also notice that, as a result of the self-duality (4.66) of $\partial^I_{mnp}$,

$$\partial^I_{mnp}\partial^{mnp\,J} = 0\,, \qquad \partial^I_{mnr}\partial^{J\,pqr} = \delta_{[m}{}^{[p}\partial^{q]rs\,J}\partial^I_{n]rs}\,. \tag{4.68}$$

Finally, let us comment on a subtlety with the 'reality' condition on the wave functions (3.9) and (3.10) and corresponding equations of motion which one should have in mind dealing with the $D=6$ case. The spinor $\lambda_{ai}$ is simplectic Majorana-Weyl, but it is not real. For such a spinor the complex conjugation condition looks as follows

$$(\lambda^i_a)^* := \bar{\lambda}_{\dot{a}i} = B_{\dot{a}}{}^b\epsilon_{ij}\lambda^j_b\,. \tag{4.69}$$

The matrix $B$ is defined by the conditions

$$B\gamma^m B^{-1} = (\gamma^m)^*\,, \qquad B\,B = -1\,, \qquad B^\dagger\,B = 1 \tag{4.70}$$

and $*$ and $\dagger$ denote, respectively, the complex and hermitean conjugation.

We note that the matrix $B_{\dot{a}}{}^b$ can be used to convert the dotted indices (of the complex conjugate representation) into undotted ones, so that one can always deal with only undotted indices, but there is no $\mathrm{SU}^*(4)$ invariant tensor for rising the $\mathrm{SU}^*(4)$ spinor indices. Thus, the spinors $\lambda^a_i$ and $\lambda_{ai}$ have different $\mathrm{SU}^*(4)$ chiralities, one of them is chiral (Weyl) and another one is antichiral.

The fields $b(X)$ and $f_{ai}(X)$ and their equations of motion considered in the next subsection are self-conjugate under the complex conjugation rules (4.69) and (4.70). In this regard we can call them 'real', since as in the $D=4$ and $D=10$ cases, they lead to real integer higher spin field equations and to simplectic Majorana-Weyl spinor equations for the half-integer spin fields.

### 4.4.2 Field equations

The equation (3.12) takes the form

$$\partial_{ai\,bj} f_{ck} - \partial_{ai\,ck} f_{bj} = 0 \,. \tag{4.71}$$

One can project (4.71) on the basis $\{\epsilon, \tilde{\tau}_I\} := \{\epsilon^{jk}, \tau_I^{jk}\}$ ($\{\epsilon, \tau_I\} := \{\epsilon_{jk}, \tau_{I\,jk}\}$) of complex $2\times 2$ matrices, which gives the following system of equations (notice that $\epsilon\epsilon = I$, $\epsilon\tilde{\tau}_I = -\tau_I\epsilon$)

$$\partial_m \left(\gamma^m_{a(b}\, f_{c)}\right) + \partial^I_{mnp} \left(\tau_I \epsilon\, \gamma^{mnp}_{a(b}\, f_{c)}\right) = 0 \,, \tag{4.72}$$

$$\partial_m \left(\tau_J\, \epsilon\, \gamma^m_{a[b}\, f_{c]}\right) - \partial^I_{mnp} (\tau_I \tilde{\tau}_J \gamma^{mnp}_{a[b}\, f_{c]}) = 0 \,. \tag{4.73}$$

The projection of these equations on the basis $\{\tilde{\gamma}^{bc}_m, \tilde{\gamma}^{bc}_{mnp}\}$ of complex $4\times 4$ matrices results in the system

$$\left(\gamma^m \partial_m + \tau_I \epsilon\, \gamma^{mnp} \partial^I_{mnp}\right) \tilde{\gamma}_{qrs} f \,= 0 \,, \tag{4.74}$$

$$\left(\tau_J \epsilon \gamma^m \partial_m - \tau_I \tilde{\tau}_J \gamma^{mnp} \partial^I_{mnp}\right)\; \tilde{\gamma}_q \; f \,= 0 \,, \tag{4.75}$$

which is thus strictly equivalent to (4.71). Contracting (4.75) with $\tilde{\gamma}^q$ and using $\tilde{\gamma}^q \gamma^{n_1 n_2 n_3} \tilde{\gamma}_q \equiv 0$ one finds that the field $f_{ai}$ obeys the Dirac equation

$$\tilde{\gamma}^m \partial_m f(x,y) = 0 \,. \tag{4.76}$$

Taking into account eq. (4.76) and the identities (4.62) one writes eqs. (4.74) and (4.75) as

$$\left(\gamma_{[m_1 m_2} \partial_{m_3]} - 2\,(\tau_I \epsilon)\, \partial^I_{m_1 m_2 m_3} - 3!\,(\tau_I \epsilon) \gamma^n{}_{[m_1} \partial^I_{m_2 m_3]n}\right) f \,= 0 \,, \tag{4.77}$$

$$\left(\tau_J \epsilon \partial_m - 3\, \tau_I \tilde{\tau}_J \gamma^{np} \partial^I_{mnp}\right)\; f \,= 0 \,, \tag{4.78}$$

Contracting eq. (4.78) with $\tau_J\epsilon$ and using the identities $\tau_I \tilde{\tau}_I = -3$, $\tilde{\tau}_I \tau_J \tilde{\tau}_I = -\tilde{\tau}_J$ one finds

$$(\partial_m - \tau_I \epsilon\, \gamma^{np} \partial^I_{mnp}\,)\, f = 0 \,. \tag{4.79}$$

Multiplying this by $\tau_J$ and using $\tau_I\epsilon = -\epsilon\tilde{\tau}_I$ one finds $(\epsilon\tilde{\tau}_J \partial_m - \tau_J \tilde{\tau}_I \gamma^{np} \partial^I_{mnp}\,)\, f = 0$ which, together with (4.78), implies

$$\tau_J \tilde{\tau}_I \gamma^{np} \partial^I_{mnp} f = -3 \gamma^{np} \partial^J_{mnp} f \qquad \Leftrightarrow \qquad \tau_I \tilde{\tau}_J \gamma^{np} \partial^I_{mnp} f = \gamma^{np} \partial^J_{mnp} f \,. \tag{4.80}$$

The consistency of (4.80) can be easily checked by contracting (any of its forms) with $\tilde{\tau}_J$. This is satisfied identically. Using (4.80) one finds from eq. (4.78) that

$$(\, \tau_I \epsilon\, \partial_m - 3 \gamma^{np} \partial^I_{mnp})\, f(x,y) = 0 \,. \tag{4.81}$$

Eq. (4.81) comprises the original eq. (4.78) and all its consequences. On the other hand, multiplying eq. (4.81) and using eqs. (4.76) and (4.67), one finds

$$(\tau_J \epsilon\, \tilde{\gamma}_{[m}\partial_{n]} + 3!\, \tilde{\gamma}^p \partial^J_{mnp}) f(x,y) = 0\,. \tag{4.82}$$

This is an equivalent form of (4.81)[14] which is most useful for the analysis below.

Now let us turn to eq. (4.77). Contracting it with $\tilde{\gamma}^{m_3}$ and using the second identity in (4.67) one finds

$$(\tilde{\gamma}_{[m}\partial_{n]} + 2\tau_I \epsilon\, \tilde{\gamma}^p \partial^J_{mnp}) f(x,y) = 0\,, \tag{4.83}$$

which can also be obtained multiplying (4.79) by $\tilde{\gamma}_n$ and antisymmetrizing the indices $m$ and $n$. On the other hand, multiplying eq. (4.77) by $\tilde{\gamma}_{m_4}$, antisymmetrizing the indices and using the third identity in (4.67) one finds

$$(\tilde{\gamma}_{[m_1 m_2 m_3}\partial_{m_4]} + 4\, \tau_I \epsilon \tilde{\gamma}_{[m_1} \partial^I_{m_2 m_3 m_4]}) f(x,y) = 0\,. \tag{4.84}$$

This equation is dual to (4.83). Indeed, multiplying (4.84) by $\epsilon^{n_1 n_2 m_1 \ldots m_4}$ and using the self-duality of the derivative $\partial^I_{[m_1 m_2 m_3]}$ and of $\tilde{\gamma}_{m_1 m_2 m_3} := (\tilde{\gamma}_{m_1 m_2 m_3})^{ab}$ (not to be confused with $\gamma_{m_1 m_2 m_3} := (\gamma_{m_1 m_2 m_3})_{ab}$ which is anti-self-dual) one arrives at (4.83). Thus one concludes that eq. (4.77) is not independent and that eqs. (4.74)-(4.75) are equivalent to the system of two equations (4.76) and (4.82), namely

$$\tilde{\gamma}^m \partial_m f(x,y) = 0\,, \tag{4.85}$$

$$(\tau^I \epsilon \tilde{\gamma}_{[m}\partial_{n]} + 3!\, \tilde{\gamma}^p \partial^I_{mnp}) f(x,y) = 0\,. \tag{4.86}$$

One more consequence of these equations is useful

$$(\gamma_{[m_1 m_2}\partial_{m_3]} + 4\, \tau_I \epsilon \partial^I_{m_1 m_2 m_3}) f(x,y) = 0\,. \tag{4.87}$$

It can be obtained by comparing (4.77) with the result of the contraction of (4.84) with $\gamma^{m_4}$. Indeed, such a comparison results in $\tau_I \epsilon \gamma^n{}_{[m_1} \partial^I_{m_2 m_3]n} f = -\tau_I \epsilon \partial^I_{m_1 m_2 m_3} f$ whose substitution in (4.77) gives (4.87).

The system of the field equations for the tensorial space scalar $b(x,y)$ originating from the $n=8$, $D=6$ version of eq. (3.11) consists of

$$\partial^p \partial_p\, b(x,y) = 0\,, \tag{4.88}$$

$$\partial^p \partial^I_{mnp}\, b(x,y) = 0 \qquad \Leftrightarrow \qquad \partial_{[m}\partial^I_{npq]}\, b(x,y) = 0\,, \tag{4.89}$$

$$\begin{gathered} \left( \partial^{I\ m_1 m_2 p}\, \partial^J_{n_1 n_2 p} - \frac{i\epsilon^{IJK}}{(3!)} \partial_{[n_1} \partial^K_{n_2]}{}^{m_1 m_2} + \frac{\delta^{IJ}}{(3!)^2} \delta^{[m_1}_{[n_1}\, \partial^{m_2]} \partial_{n_2]} \right) b(x,y) = 0 \quad \Leftrightarrow \\ \Longleftrightarrow \quad \begin{cases} \left( \partial^{m_1 m_2 p\, (I}\, \partial^{J)}_{n_1 n_2 p} + \frac{1}{(3!)^2} \delta^{IJ} \delta^{[m_1}_{[n_1}\, \partial^{m_2]} \partial_{n_2]} \right) b(x,y) = 0 \\ \left( \partial^{m_1 m_2 p\, [I}\, \partial^{J]}_{n_1 n_2 p} - \frac{i\epsilon^{IJK}}{(3!)} \partial_{[n_1} \partial^K_{n_2]}{}^{m_1 m_2} \right) b(x,y) = 0 \end{cases} . \end{gathered} \tag{4.90}$$

One more useful equation is

$$\left( \delta^{[n_1 n_2}_{[m_1 m_2} \partial_{m_3]} \partial^{n_3]} + 8\, \partial^I_{m_1 m_2 m_3} \partial_I^{n_1 n_2 n_3} \right) b = 0\,. \tag{4.91}$$

[14] To check this one multiplies (4.82) by $\gamma^n$ and uses the Dirac equation (4.76).

A simple way to obtain eqs. (4.88), (4.89) and (4.90) is to observe that the derivative $\partial_{\gamma\,\delta} b$ of the bosonic field $b(x,y)$ obeying the bosonic equation (3.11) can be treated as a set of solutions of the fermionic equation (3.12) (the extra spinor index $\delta$ being regarded as the label of the fermion-like solutions). Then in view of the form of eqs. (4.85) and (4.86) on finds that the independent bosonic equations following from (3.11) are

$$\tilde{\gamma}^m \partial_m\,(\epsilon\cdot\gamma^n\,\partial_n\,+\,\tau_I\cdot\gamma^{npq}\,\partial^I_{npq})b(x,y)=0\,, \tag{4.92}$$

$$(\,\tau_I\epsilon\cdot\tilde{\gamma}_{[m_1}\partial_{m_2]}+\,3!\,\tilde{\gamma}^{m_3}\partial^I_{m_1m_2m_3})\,(\epsilon\cdot\gamma^n\,\partial_n\,+\,\tau_J\cdot\gamma^{npq}\,\partial^J_{npq})b(x,y)=0\,. \tag{4.93}$$

Now obsereve that the terms in (4.92) proportional to $\epsilon_{ij}$ and $\tau_{I\,ij}$ should vanish separately. They produce, respectively, the Klein-Gordon equation (4.88) and eq. (4.89) (to derive the latter one should remember that $\tilde{\gamma}^{m_1\ldots m_4}\;=\;1/2\epsilon^{m_1\ldots m_4pq}\tilde{\gamma}_{pq}$; not to be confused with $\gamma^{m_1\ldots m_4}=(\tilde{\gamma}^{m_1\ldots m_4})^T=-1/2\epsilon^{m_1\ldots m_4pq}\gamma_{pq}$ where the sign is opposite). With this in mind we find that the only independent part of eq. (4.93) is

$$\mathrm{tr}\left[\tau_J\epsilon\cdot\gamma_{n_1n_2}(\,\tau_I\epsilon\cdot\tilde{\gamma}_{[m_1}\partial_{m_2]}+\,3!\,\tilde{\gamma}^{m_3}\partial^I_{m_1m_2m_3})\,(\epsilon\cdot\gamma^n\,\partial_n\,+\,\tau_K\cdot\gamma^{npq}\,\partial^K_{npq})\right]\;b(x,y)=0\,, \tag{4.94}$$

which gives eq. (4.90).

Alternatively, as in $D=4$ and $D=10$, instead of direct computations one can obtain the field equations of $b(X)$ and $f_\alpha(X)$ by using the plane wave solution (3.5)

$$\Phi(x,\lambda)=e^{i\frac{1}{16\cdot 3!}(\lambda\tilde{\gamma}_{mnp}\tau^I\lambda)\,y^{mnp}_I}\;e^{i\frac{1}{8}(\lambda\tilde{\gamma}_m\epsilon\lambda)\,x^m}\phi(\lambda)$$

and Fierz identities for the $D=6$ $\gamma$-matrices.

To analyse the consequences of eqs. (4.85)–(4.91) in the effective $D=6$ higher spin field theory we expand $b(X)$ and $f_\alpha(X)$ in series of $y^{[3]}_I$

$$b(x,\,y)\,=\,\phi(x)+y^{[3]}_I F^I_{[3]}(x)+y^{[3]_1}_{I_1}\,y^{[3]_2}_{I_2}\,\hat{R}^{I_1I_2}_{[3]_1[3]_2}(x)+\sum_{s=3}^{\infty}\,y^{[3]_1}_{I_1}\cdots y^{[3]_s}_{I_s}\,\hat{R}^{I_1\cdots I_s}_{[3]_1\,\cdots\,[3]_s}(x)\,,$$

$$f_\alpha(x,\,y)\,=\,\psi_\alpha(x)+y^{[3]}_I\hat{\mathcal{R}}^I_{\alpha\,[3]}(x)+\sum_{s=5/2}^{\infty}\,y^{[3]_1}_{I_1}\cdots y^{[3]_{s-1/2}}_{I_{s-1/2}}\,\hat{\mathcal{R}}^{I_1\cdots I_{s-1/2}}_{\alpha\,[3]_1\cdots\,[3]_{s-1/2}}(x)\,.$$

As in $D=4$, and 10, the equation (4.82) relates the gamma-trace of the spin $s$ spinor-tensor to the first derivative of the spin $s-1$ spinor-tensor, *e.g.*

$$\tilde{\gamma}^p\hat{\mathcal{R}}^I_{mnp}=-\frac{1}{3!}\partial_{[m}\tilde{\gamma}_{n]}\tau^I\psi\,. \tag{4.95}$$

A novelty of the $D=6$ case is that in virtue of eq. (4.87)also the 'tau-trace' of the spin $s$ spinor-tensor is related to the first derivative of the spin $s-1$ spinor-tensor, *e.g.*

$$\tau_I\epsilon\hat{\mathcal{R}}^I_{mnp}=-\frac{1}{4}\partial_{[m}\gamma_{np]}\psi\,. \tag{4.96}$$

Thus the gamma-traces and the tau-traces of the spinor-tensors do not correspond to independent physical degrees of freedom.

The equation (4.90) relates the trace of the spin $s$ field strength to the first derivative of the spin $s-1$ and to the second derivative of the spin $s-2$ field strength. For instance,

$$\hat{R}^{I_1 I_2}_{m_1 n_1 p, m_2 n_2}{}^{p} = \frac{i}{12}\epsilon^{I_1 I_2 I_3}\partial_{[m_2} F^{I_3}_{n_2] m_1 n_1} - \frac{1}{2(3!)^2}\delta^{I_1 I_2}\partial_{[m_1}\eta_{n_1][m_2}\partial_{n_2]}\phi\,. \tag{4.97}$$

Eq. (4.91) relates the SO(3) trace of the spin-$s$ field strength to the second derivatives of the spin $s-2$ field strength, for example

$$\delta_{I_1 I_2}\hat{R}^{I_1 I_2}_{m_1 n_1 p_1,}{}^{m_1 n_2 p_2} = -\frac{1}{16}\delta^{[m_2 n_2}_{[m_1 n_1}\partial_{p_1]}\partial^{p_2]}\phi\,. \tag{4.98}$$

The transversality of $\hat{R}^{I_1\cdots I_s}_{[3]_1\,\cdots\,[3]_s}(x)$ and $\hat{\mathcal{R}}^{I_1\cdots I_{s-1/2}}_{\alpha\,[3]_1\cdots\,[3]_{s-1/2}}(x)$ is the consequence of eq. (4.89). Their self-duality in each set of antisymmetric indices is automatic. The (gamma) traceless parts $R^{I_1\cdots I_s}_{[3]_1\,\cdots\,[3]_s}(x)$ and $\mathcal{R}^{I_1\cdots I_{s-1/2}}_{\alpha\,[3]_1\cdots\,[3]_{s-1/2}}(x)$ of the (spinor)-tensors describe propagating higher spin degrees of freedom corresponding to $Spin(1,5)$ irreps characterized by rectangular Young diagrams with three rows of equal length. They are the field strengths of the gauge fields characterized by rectangular Young diagrams with two rows. All this implies that the propagating fields carry irreps of the conformal group $Spin(2,6)$.

A new feature of the $D=6$ case is the degeneracy of these irreps due to the internal SO(3) symmetry. The GL$(6,\mathbb{R})$ irreducibility implies symmetry under the exchange of multi-indices. This property, along with the commutativity of $y^{[3]}_I$, implies that the field strengths $R^{I_1\ldots I_s}_{[3]_1\cdots[3]_s}$ and $\mathcal{R}^{I_1\cdots I_{s-1/2}}_{\alpha\,[3]_1\cdots\,[3]_{s-1/2}}(x)$ are also symmetric in the internal SO(3) indices $I$. This leads to the fact that each propagating field corresponds to a spin-$s$ irrep of SO(3), thus the degeneracy of the spin-$s$ irrep is equal to $2[s]+1$.

In other words, the quantum spectrum of the tensorial $n=8$ superparticle is formed by an infinite number of conformally invariant (self-dual) "multi-3-form" higher spin fields in $D=6$ whose number for each value of spin $s$ is $2[s]+1$, and which form the $(2[s]+1)$-dimensional representation of the group SO(3). This differs from the cases of $n=4,\ D=4$ and $n=16,\ D=10$, where the conformal fields of each spin $s$ appear in the quantum spectrum only once.

## 5. Conclusion

In this paper we have analyzed the geometrical structure of conformally invariant higher spin fields and have shown that in $D=3,4,6$ and 10 space-time the massless conformal higher spin fields arise as a result of the quantization of the dynamics of the twistor-like particle, respectively, in $n=2,4,8,$ and 16 tensorial space. The $D=3$ and $D=4$ cases have already been considered in the literature, while the $D=6$ and $D=10$ results are new.

In each of these cases, the infinite sum of irreps of the conformal group $Spin(2,D)$ gets combined into an infinite-dimensional representation of the supergroup OSp$(1|2n,\mathbb{R})$ (with $n=2(D-2)$). The latter is associated with the solutions of the OSp$(1|2n,\mathbb{R})$-invariant scalar and spinor field equation in tensorial space. The superfield form of these equations, both in flat tensorial superspace and on the supergroup manifold OSp$(1|n,\mathbb{R})$

was constructed in [58]. When reduced to the effective $D = 4$, 6 and 10 space-time the tensorial equations give rise, in a very natural way, to geometric conformal higher spin field equations in the Bargmann-Wigner form.

To conclude let us discuss possible directions in which present work might be developed:

- One of them is to generalize above results to the $AdS_D$ space whose tensorial extension is the group manifold $\mathrm{Sp}(n,\mathbb{R})$ [11, 5, 6, 7]. For example, one may take as the starting point the wave function [7]

$$\Phi(X^{\alpha\beta},\lambda) = \int d^n y\, \sqrt{\det G^{-1}(X)}\, e^{iX^{\alpha\beta}(\lambda_\alpha + \frac{1}{8r}y_\alpha)(\lambda_\beta+\frac{1}{8r}y_\beta)+i\lambda_\alpha y^\alpha}\, \widetilde{\varphi}(y)\,, \qquad (5.1)$$

  where $G^{-1\beta}_\alpha(X) = \delta^\beta_\alpha + \frac{1}{4r}X_\alpha{}^\beta$, and $r$ is the AdS radius, and derive the $\mathrm{Sp}(n,\mathbb{R})$ analog of the field equations (3.11)–(3.12) [7]

$$\nabla_{\alpha[\beta}\nabla_{\gamma]\delta} b = \frac{1}{16r}\left(C_{\alpha[\beta}\nabla_{\gamma]\delta} - C_{\delta[\gamma}\nabla_{\beta]\alpha} + 2C_{\beta\gamma}\nabla_{\alpha\delta}\right) b + \\ + \frac{1}{64r^2}\left(2C_{\alpha\delta}C_{\beta\gamma} - C_{\alpha[\beta}C_{\gamma]\delta}\right) b\,, \qquad (5.2)$$

$$\nabla_{\alpha[\beta} f_{\gamma]} = -\frac{1}{4r}\left(C_{\alpha[\gamma}f_{\beta]} + 2C_{\beta\gamma}f_\alpha\right)\,, \qquad (5.3)$$

  where $\nabla_{\alpha\beta} \equiv G^{-1\gamma}_\alpha(X) G^{-1\delta}_\beta(X)\partial_{\gamma\delta}$. However, to reduce the tensorial $\mathrm{Sp}(n,\mathbb{R})$ model to the higher spin field theory on $AdS_D$ by disentangling the $x^m$ and $y^{mn\cdots q}$ dependence is much more cumbersome problem than in flat tensorial space due to the complicated $X^{\alpha\beta}$ dependence of the plane wave solution $\Phi(X,\lambda)$ and of the covariant derivative $\nabla_{\alpha\beta}$. To this end one may also try to use other realizations of the $\mathrm{Sp}(n,\mathbb{R})$ model considered in [5, 6], or its twistor counterpart constructed in [54].

- It would be also interesting to obtain the $n = 8,\ D = 6$ spectrum by expanding the wave function in $\lambda_\alpha$. The two-component quaternionic formalism can be useful for this purpose [55, 56], like the Weyl spinor formalism for the case $n = 4,\ D = 4$. This can provide a new realization of the $\mathrm{OSp}(1|16,\mathbb{R})$ infinite-dimensional irreducible representations.

- The reduction of the $D = 6$ model to $D = 5$ produces an infinite tower of completely symmetric gauge fields of all spins with exactly identical internal SO(3) structure for a given spin $s$, as can be easily seen. In analogy with Hull's conjecture [52], a strong coupling limit of a hypothetical $D = 5$ interacting higher-spin theory might be expected to be an interacting $D = 6$ exotic theory whose free limit is the $n = 8$ tensorial model considered in this paper. Though appealing, such a scenario seems to be difficult to realize. Indeed, switching on interactions is still a challenging open problem for gauge fields which are either higher-spin, chiral, or of mixed symmetry[15] (especially when they possess *all* these properties simultaneously). Note that $AdS_5$ and $AdS_7$ twistor counterparts of the tensorial model have recently been discussed in [54].

[15] Some no-go theorems have recently been proved for chiral form and two-column field self-interactions (see, respectively, [57] and [38], and refs. therein).

- A possibility of introducing non-linearity in higher spin field equations directly in a curved tensorial superspace was analyzed in [58]. It was shown that if one tries to maintain manifest conformal invariance also in the non-linear theory this puts too severe restrictions on the geometry of the tensorial superspace and does not lead to higher spin interactions. This indicates that a possible way out might be related to breaking conformal symmetry.

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## A. Useful gamma matrix identities

All antisymmetrizations of indices are denoted by brackets [ ] and have unit weight. All symmetrizations of indices are denoted by brackets ( ) and have unit weight. Some of the identities presented here are taken from [59].

**Any dimension D.** The Clifford algebra is

$$\gamma_m \gamma_n + \gamma_n \gamma_m = 2\,\eta_{mn}\,.$$

The matrices

$$\gamma_{m_1 \ldots m_p} \equiv \gamma_{[m_1} \cdots \gamma_{m_p]}$$

satisfy the orthonormality relations

$$tr[\gamma_{m_1 \ldots m_p} \gamma^{n_1 \ldots n_q}] = (-)^{p(p-1)/2}\; p!\; \delta^{n_1 \ldots\, n_q}_{m_1 \ldots m_p}\, \delta_{pq}\, tr[1]\,, \tag{A.1}$$

where the Kronecker symbols are defined by

$$\delta^{n_1 \ldots\, n_p}_{m_1 \ldots m_p} \equiv \delta^{[n_1}_{m_1} \ldots \delta^{n_p]}_{m_p} = \delta^{n_1}_{[m_1} \ldots \delta^{n_p}_{m_p]}\,. \tag{A.2}$$

$$\begin{aligned} \gamma_{m_1 \ldots m_{p+1}} &= \gamma_{m_1 \ldots m_p} \gamma_{m_{p+1}} \;-\; p\;\gamma_{[m_1 \ldots m_{p-1}}\,\eta_{m_p] m_{p+1}}\,, \\ \gamma_n \gamma_{m_1 \ldots m_p} &= (-)^p\, \gamma_{m_1 \ldots m_p} \gamma_n \;+\; 2p\;\eta_{n[m_1} \gamma_{m_2 \ldots m_p]}\,, \end{aligned} \tag{A.3}$$

or in general

$$\gamma^{n_1 \cdots n_i}\, \gamma_{m_1 \cdots m_j} = \sum_{k=0}^{k=min(i,j)} \frac{i!\,j!}{(i-k)!\,(j-k)!\,k!}\, \gamma^{[n_1 \cdots n_{i-k}}{}_{[m_{k+1} \cdots m_j}\, \delta^{n_i}_{m_1}\, \delta^{n_{i-1}}_{m_2} \cdots \delta^{n_{n-k+1}]}_{m_k]}\,. \tag{A.4}$$

We also use the following identity

$$\begin{aligned} \gamma^m\, \gamma^{n_1 \cdots n_q}\, \gamma_m &= (-)^q\, (D-q)\, \gamma^{n_1 \cdots n_q}\,, \\ \gamma^{n_1 n_2 n_3}\, \gamma^m\, \gamma_{n_1 n_2 n_3} &= -(D-6)\,(D-1)\,(D-2)\, \gamma^m\,. \end{aligned} \tag{A.5}$$

**D = 3, 4, 6, 10.** These dimensions are respectively associated to the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ [55]. A common property of the gamma matrices considered in this paper is the Fierz identity

$$(\gamma^m)_{\alpha(\beta}(\gamma_m)_{\gamma\delta)} = 0\,. \tag{A.6}$$

**D = 6.**

$$\gamma^m_{ab}\tilde\gamma^{cd}_m = -4\delta_{[a}{}^c\delta_{b]}{}^d\,, \qquad \gamma^m_{ab} = \frac{1}{2}\epsilon_{abcd}\tilde\gamma^{m\,cd}\,, \tag{A.7}$$

$$(\gamma_m)_{ab}(\gamma^m)_{cd} = -2\epsilon_{abcd}\,, \tag{A.8}$$

from which originates

$$(\gamma_m)_{a(b}(\gamma^m)_{c)d} = 0 \tag{A.9}$$

The matrices $\tau_I := \tau_{I\,ij} = \epsilon_{jj'}\,\sigma_{I\,i}{}^{j'}$ and $\tilde\tau_I := \tau_I^{ij} = \epsilon^{ii'}\,\sigma_{I\,i'}{}^j$ $(I = 1,2,3)$ obey

$$\tilde\tau_I\,\tau_J + \tilde\tau_J\,\tau_I = -2\delta_{IJ} \quad\Leftrightarrow\quad \tau_I^{ij'}\,\tau_{Jj'j} + \tau_J^{ij'}\,\tau_{Ij'j} = -2\delta_{IJ}\delta_j{}^i\,, \tag{A.10}$$

$$\tau_I^{ii'}\,\tau_{I\,jj'} = -2\delta_{[j}{}^i\delta_{j']}{}^{i'} \quad\Leftrightarrow\quad \tau_{I\,ii'}\tau_{Ijj'} = 2\epsilon_{i(j}\epsilon_{j')i'}\,. \tag{A.11}$$

The appearance of Pauli matrices $\sigma_I := \sigma_{I\,i}{}^j$ $(I = 1,2,3)$ is reminiscent of the $D = 6$ quaternionic structure

$$\sigma_I\,\sigma_J = \delta_{IJ} + i\epsilon_{IJK}\sigma_K\,.$$

The antisymmetric spin-tensor $\epsilon_{ij}$ and its inverse $\epsilon^{ij}$ are used to lower and to rise isospinorial SU(2) indices, see above and

$$\epsilon^{ik}\epsilon_{kj} = \delta^i_j\,, \qquad \tau_{I\,ij} = \epsilon_{ii'}\epsilon_{jj'}\tau_I^{i'j'}\,.$$

**D = 10.** The set of $16\times 16$ symmetric matrices with respect to the pair of lower indices $\alpha\beta$ is given by $\gamma^m_{\alpha\beta}$ and by $\gamma^{m_1\ldots m_5}_{\alpha\beta}$ which are self-dual in spacetime indices

$$\gamma^{m_1\ldots m_5}_{\alpha\beta} = \frac{1}{5!}\,\epsilon^{m_1\ldots m_5 n_1\ldots n_5}\gamma_{n_1\ldots n_5\,\alpha\beta}\,.$$

In contrast, $\gamma^{m_1\ldots m_5\,\alpha\beta}$, are anti-self-dual

$$\gamma^{m_1\ldots m_5\;\alpha\beta} = -\frac{1}{5!}\,\epsilon^{m_1\ldots m_5 n_1\ldots n_5}\gamma_{n_1\ldots n_5}{}^{\alpha\beta}\,.$$

As a result

$$\mathrm{tr}(\gamma^{m_1\ldots m_5}\,\tilde\gamma_{n_1\ldots n_5}) := \gamma^{m_1\ldots m_5}_{\alpha\beta}\gamma^{\beta\alpha}_{n_1\ldots n_5} = 16\cdot 5!\left(\delta_{[m_1}{}^{[n_1}\ldots\delta_{m_5]}{}^{m_5]} + \frac{1}{5!}\,\epsilon_{m_1\ldots m_5}{}^{n_1\ldots n_5}\right),$$

in distinction to $D \neq 10$ (for $D > 5$) where only the first term is present.

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